M.Sc
Part - II

Paper IV - VI
Optional - I

MATHEMATICAL

MATHEMATICS

STATISTICS

Original Notes

Traders
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HOUSE OF
QUALITY
NOTES

"MATHEMATICAL STATISTICS"

Course Outline

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Matric, F.Sc., B.A/B.Sc.
O & A - Level M.A. (Eng. Eco.)
B.Sc. (Maths, Phy.)
Ph: 051-4421542, 0333-5126581

CH # 1 :- Measures of Central Tendency
CH # 2 :- Measures of Dispersion
CH # 3 :- Probability
CH # 4 :- Random Variables
CH # 5 :- Discrete Probability Distributions
1) Discrete uniform Distribution
2) Bernoulli's Distribution
3) Binomial Distribution
4) Negative Binomial Distribution
5) Hypergeometric Distribution
6) Poisson Distribution
7) Geometric Distribution
CH # 6 :- Continous Probality Distributions
1) Normal Distribution
2) Gamma Distribution
3) Beta Distribution
4) Continous uniform Distribution
5) Exponential Dist
CH # 7 :- Sampling Distributions
1) Chi-Square Distributions
2) t-distribution
3) F-Distribution.

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CH # 8 :- Regression & Correction

PREPARED BY
SAQIB MAHMOOD

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CH # 1 :- Measures of Central Tendency

Statistics:- ① Statistics is science of Average.

② Statistics is science of Counting.

③ Statistics is science of Estimations & Probabilities.

Data:- The Raw material of facts & figure is called data.

Information:- Data when Processed becomes information

OR

Useful data is called Information.

Note:- Central Tendency is also called Average.

Average:- Average is a single value which is calculated to represent whole data.

Types of Averages

1- Arithmatic Mean
2- Median 3- Mode
4- Geometeric Mean
5- Hormonic Mean.

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1- Arithmetic Mean

If $x_1, x_2, \ldots, x_n$ are values of given data then Arithmetic Mean OR Mean is denoted as $\bar{x}$ & is calculated as,

$$\bar{x} = \frac{\Sigma x}{n}$$

where

Σ represents sum of values

& n = Number of values

Remark:-

Data

→ Group Data

→ UnGrouped Data

In Case of ungrouped Data,

$$\bar{x} = \frac{\Sigma x}{n}$$

In Case of Grouped Data

$$\bar{x} = \frac{\Sigma f x}{\Sigma f}$$

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Example :- Calculate the Arithmetic Mean of given data (ungrouped)

65, 60, 64, 69, 70, 75

Solution :- As we know

$$\bar{X} = \frac{\Sigma x}{n}$$

$$\bar{X} = \frac{65+60+64+69+70+75}{6}$$

$$\bar{X} = 67.16$$

Example :- Calculate the Arithmetic Mean of given data (Grouped)

Income	f
35 – 39	13
40 – 44	15
45 – 49	28
50 – 54	17
55 – 59	12
60 – 64	10
65 – 69	5

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Solution :- As we know that

$$\bar{X} = \frac{\Sigma f x}{\Sigma f}$$

Income	F	x	fx
35 – 39	13	37	481
40 – 44	15	42	630
45 – 49	28	47	1316
50 – 54	17	52	884
55 – 59	12	57	684
60 – 64	10	62	620
65 – 69	5	67	335
	$\Sigma F = 100$		$\Sigma fx = 4950$

So,

$$\bar{X} = \frac{4950}{100}$$

$$\bar{X} = 49.5$$

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Advantages:-

① It is easy to calculate.

② All Entries are used.

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Disadvantages:-

It is highly Effected by Extereme values.

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Arithmetic Mean

Combined Mean — Weighted Mean.

Combined Arithmetic Mean:- If $\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_k$ be Arithmetic Mean of k distributions with respect to frequencies $n_1, n_2, \ldots, n_k$ then Combined Arithmetic Mean is denoted & defined as,

$$\bar{X}_c = \frac{n_1\bar{X}_1 + n_2\bar{X}_2 + \cdots + n_k\bar{X}_k}{n_1 + n_2 + \cdots + n_k}$$

$$\bar{X}_c = \frac{\sum_i n_i\bar{X}_i}{\sum_i n_i}$$

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Example:- Findout Combined Mean from the following data.

	Series (X_1)	Series (X_2)
A.M	$12 = \bar{X}_1$	$20 = \bar{X}_2$
no of itmes	$80 = n_1$	$60 = n_2$

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$$\overline{X}_c = \frac{n_1\overline{X}_1 + n_2\overline{X}_2}{n_1+n_2}$$

$$\overline{X}_c = \frac{80\times12 + 60\times20}{80+60}$$

$$\overline{X}_c = \frac{960 + 1200}{140}$$

$$\overline{X}_c = \frac{2160}{140}$$

$$\overline{X}_c = 15.43$$

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Weighted Arithmetic Mean:-

Some times different observations do not have same importence. Some observations for some reasons have great importence. The relative importence is called weight (W) of the observations.

If the observations $x_1, x_2, \ldots, x_n$ have the respective weight $w_1, w_2, \ldots, w_n$ then Weight Arithmetic mean is denoted & defined as,

$$\overline{X}_w = \frac{w_1x_1 + w_2x_2 + \cdots + w_nx_n}{w_1 + w_2 + \cdots + w_n}$$

$$\overline{X}_w = \frac{\sum_i w_i x_i}{\sum_i w_i}$$

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Example:- A student find the marks in math, Phy, Eng & stat are respectively 82, 86, 90 & 70. If the respective credit recived for cources are 3, 5, 3 & 1 then Determine the Approximate Average.

Solution:-

Subject	Marks (x)	weight (w)	wx
Math	82	3	246
Phy	86	5	430
Eng	90	3	270
Stat	70	1	70
		$\Sigma w = 12$	$\Sigma wx = 1016$

As we know that

$$\bar{X}_w = \frac{\Sigma wx}{\Sigma w}$$

$$\bar{X}_w = \frac{1016}{12}$$

$$\bar{X}_w = 84.66$$

As Required.

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Assignment # 1

Question 1) The logarithms of 10 Values of X are 1.8062, 1.2304, 1.6532, 1.5798, 1.4314, 0.7782, 1.6812, 1.0414, 1.7559, 1.5315.
Calculate arthimetic Mean of X.

Solution:- As we know that

$$\bar{X} = \frac{\Sigma x}{n}$$

$$\bar{X} = \frac{1.8062+1.2304+1.6532+1.5798+1.4314+0.7782+1.6812+1.0414+1.7559+1.5315}{10}$$

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$$\bar{X} = \frac{14.4892}{10}$$

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$\bar{X} = 1.44892$ Required.

Question 2) There are three sections in B.Com first year in a certain college. The number of students in each section & average marks obtained by them in the Paper of statistics in the Anual Examination are as follows,

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Section	Average Marks	No. of student
A	75	50
B	60	60
C	50	50

Find Combined Arithmetic Mean.

Solution:- As we know

$$\bar{X}_c = \frac{n_1\bar{X}_1 + n_2\bar{X}_2 + n_3\bar{X}_3}{n_1+n_2+n_3}$$

$$\bar{X}_c = \frac{50\times75 + 60\times60 + 50\times50}{50+60+50}$$

$$\bar{X}_c = \frac{3750 + 3600 + 2500}{160}$$

$$\bar{X}_c = 61.56$$

Required.

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Question 3) Calculate Arithmetic Mean from the following Group Data.

Classes	10–19	20–29	30–39	40–49	50–59
Frequencies	5	25	40	20	10

Solution:-

classes	f	x	fx
10 — 19	5	14.5	72.5
20 — 29	25	24.5	612.5
30 — 39	40	34.5	1380
40 — 49	20	44.5	890
50 — 59	10	54.5	545
	$\Sigma f = 100$		$\Sigma fx = 3500$

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As we know

$$\bar{X} = \frac{\Sigma fx}{\Sigma f}$$

$$\bar{X} = \frac{3500}{100}$$

$$\bar{X} = 35 \text{ Required.}$$

Question 4) Show that weighted Arithmetic Mean of first n natural numbers whose weights are equal to the corresponding numbers is equal to $\frac{2n+1}{3}$

Solution :-

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25/25

Rashid

X	w	wx
1	1	1^2
2	2	2^2
3	3	3^2
⋮	⋮	⋮
n	n	n^2

As we know that

$$\bar{X}_w = \frac{\Sigma wx}{\Sigma w}$$

$$= \frac{1^2 + 2^2 + 3^2 + \cdots + n^2}{1 + 2 + 3 + \cdots + n}$$

$$= \frac{\frac{n(n+1)(2n+1)}{6}}{\frac{n(n+1)}{2}}$$

$$\bar{X}_w = \frac{2n+1}{3}$$

As Required.

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2- Geometeric Mean

If $x_1, x_2, \ldots, x_n$ be n values of data then their G.M is defined as,

$$(x_1 . x_2 , \ldots . x_n)^{1/n} = \sqrt[n]{x_1 . x_2 \ldots . x_n}$$

As $G = [x_1 . x_2 , \ldots x_n]^{1/n}$

Taking log on both sides

$$\log G = \log [x_1 . x_2 \ldots . x_n]^{1/n}$$

$$= \frac{1}{n} \log [x_1 . x_2 \ldots x_n]$$

$$= \frac{1}{n} [\log x_1 + \log x_2 + \ldots + \log x_n]$$

$$\log G = \frac{1}{n} \Sigma \log x_i$$

$$G = \text{Antilog} \left[\frac{\Sigma \log x_i}{n} \right]$$

This is formula of G.M for ungrouped data.

If the n non zero & Positive variate values $X_1, X_2, \ldots, X_n$ occure $f_1, f_2, \ldots f_n$ times respectively

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then G.M. is defined by,

$$G = \left[x_1^{f_1} \cdot x_2^{f_2} \cdots\cdots x_n^{f_n} \right]^{1/n}$$

Taking log on both sides

$$\log G = \frac{1}{n} \log \left[x_1^{f_1} \cdot x_2^{f_2} \cdots\cdots x_n^{f_n} \right]$$

$$\log G = \frac{1}{n} \left[f_1 \log x_1 + f_2 \log x_2 + \cdots\cdots + f_n \log x_n \right]$$

$$\log G = \frac{1}{n} \left[\sum_i f_i \log x_i \right]$$

$$G = \text{Antilog} \left[\frac{\sum_i f_i \log x_i}{n} \right]$$

$$G = \text{Antilog} \left[\frac{\sum_i f_i \log x_i}{n = \sum_i f_i} \right]$$

This is expressed for Grouped data.

Example 1) Find G.M of 2, 4, 8

As we know

$$G = (n_1 . n_2 . n_3)^{1/n}$$

$$= (2 \times 4 \times 8)^{1/3}$$

$$= (64)^{1/3}$$

$$= 4$$

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Example 2) Find G.M. from the following data,
9.7, 0.0009, 178.7, 0.874, 1238, 0.012, 89.9, 78.4, 0.989, 0.008

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Solution:-

X_i	$\log X_i$
9.7	0.9868
0.0009	-3.0458
178.7	2.2521
0.874	-0.0585
1238	3.0927
0.012	-1.9208
89.9	1.9538
78.4	1.8943
0.989	-0.0048
0.008	-2.0969
	$\Sigma \log X_i = 3.0529$

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$n = 10$

So,
$$G = \text{Antilog}\left[\frac{\Sigma \log X_i}{n}\right]$$
$$= \text{Antilog}\left[\frac{3.0529}{10}\right]$$
$$G = \text{Antilog}[0.3053]$$
$$G = 2.0197$$
As Required.

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Example 3) Calculate G.M from the following.

Weights (£)	Frequency
100 — 104	24
105 — 109	30
110 — 114	45
115 — 119	65
120 — 124	72

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Solution:-

weight	f	x	log x	f log x
100-104	24	102	2.0086	48.2064
105-109	30	107	2.0294	60.8820
110-114	45	112	2.0492	92.2140
115-119	65	117	2.0682	134.4330
120-124	72	122	2.0864	150.2179
	$\Sigma f = 236$			$\Sigma f \log x = 485.9562$

As we know

$$G = \text{Antilog}\left[\frac{\Sigma f \log x}{\Sigma f}\right]$$

$$= \text{Antilog}\left[\frac{485.9562}{236}\right]$$

$$= \text{Antilog}(2.0591)$$

$G = 114.58$ As Required.

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3- Harmonic Mean

It is the Ratio of the number of items & sum of Resiprocal of the items.

If the observations are $x_1, x_2, \dots, x_n$ then H.M "H" is given by

$$H = \frac{\text{no of items}}{\text{sum of items of resiprocal}}$$

$$H = \frac{n}{\Sigma(1/x)}$$

This Expression for unGrouped Data.

For Grouped Data:-

$$H = \frac{f_1 + f_2 + \dots + f_n}{\frac{f_1}{x_1} + \frac{f_2}{x_2} + \dots + \frac{f_n}{x_n}}$$

$$H = \frac{\Sigma f}{\Sigma(f/x)}$$

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Example 1)

Compute H.M of

85, 45.4, 52, 40, 129.5, 14.5, 22, 260.

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Solution:-

X	1/X
85	0.0118
45.4	0.0220
52	0.0200
40	0.0250
129.5	0.0077
14.5	0.0690
22	0.0455
260	0.0038

$\Sigma(1/x) = 0.2048$

As,

$$H = \frac{n}{\Sigma(1/x)}$$

$$H = \frac{8}{0.2048}$$

$H = 39.0625$ As Required.

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Example 2)

Calculate H.M from the following Data.

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X	f
2.1	27
2.4	42
2.7	59
3.0	69
3.3	54
3.6	43
3.9	21
4.2	16
4.5	9

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Solution :-

X	f	f/x
2.1	27	12.8571
2.4	42	17.5000
2.7	59	21.8519
3.0	69	23.0000
3.3	54	16.3636
3.6	43	11.9444
3.9	21	5.3846
4.2	16	3.8095
4.5	9	2.0000
	$\Sigma f = 340$	$\Sigma f/x = 114.7111$

So, $H = \frac{\Sigma f}{\Sigma f/x} = \frac{340}{114.7111}$

$H = 2.96$ As Required.

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Question:- Find G.M of the Series.

1, 2, 4, 8, 16, ..., 2^n

Solution :-

$$G = (1\times2\times4\times8\times16\times\cdots\times2^n)^{1/n}$$

$$G = [2^0\times2^1\times2^2\times2^3\times2^4\times\cdots\times2^n]^{1/n}$$

$$G = [2^{1+2+3+\cdots n}]^{1/n}$$

$$G = [2^{\frac{n(n+1)}{2}}]^{1/n}$$

$$G = 2^{\frac{n+1}{2}}$$ As Required.

Question:- Show that A.M of the series

1, 2, 4, 8, 16, ... 2^n is $\frac{2^{n+1}-1}{n+1}$

Solution:- As we know

$$A.M = \frac{\Sigma X}{n}$$

Here $n = n+1$

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$\Sigma X = 1+2+4+8+16+ \dots + 2^n$

It is Geometeric Series

So,

$$S_n = \frac{a(r^n - 1)}{r-1}$$

$a = 1, \quad r = \frac{2}{1} = 1, \quad n = n+1$

$$S_n = \frac{1\,(2^{n+1} - 1)}{2-1}$$

$$S_n = 2^{n+1} - 1$$

So,

$$\Sigma X = 2^{n+1} - 1$$

Also,

$$A.M = \frac{\Sigma X}{n}$$

$$A.M = \frac{2^{n+1} - 1}{n+1}$$

A.S Required.

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4- The Mode

Mode is defined as the value in the data which occures the greatest number of items provided such a value exist.

e.g,

2, 5, 6, 7, 2, 2, 4, 5, 7

Mode = 2

is called unimode.

2, 5, 6, 7, 6, 7, 4, 3, 1

Mode = 6, 7

is called unimode.

1, 5, 9, 6, 7, 3, 2, 10

Mode does not Exists.

It is

For UnGrouped Data.

Grouped Data: Mode can be calculated as,

$$\text{Mode} = l + \frac{(f_m - f_1)}{(f_m - f_1) + (f_m - f_2)} \times h$$

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where,

l = lower class boundry of Model class.
f_m = frequency of model class

f_1 = frequency associated with the class preceding the model class.

f_2 = frequency associated with the class following the model class.

h = length of class interval.

Example:- Calculate Mode from the following Data.

Height	f
48 – 50	5
50 – 52	7 → f_1
52 – 54	10 → f_m
54 – 56	9 → f_2
56 – 58	6
58 – 60	3

l (lower boundary of 52 – 54)

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Solution:- $l = 52$, $f_1 = 7$, $f_2 = 9$
$f_m = 10$, $h = 54 - 52 \Rightarrow 2$

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So,

$$\text{Mode} = l + \frac{(f_m - f_1)}{(f_m - f_1) + (f_m - f_2)} \times h$$

$$= 52 + \frac{(10-7)}{(10-7)+(10-9)} \times 2$$

$$= 52 + \frac{3}{3+1} \times 2$$

$$= 52 + \frac{6}{4}$$

$$= 52 + \frac{3}{2}$$

$$= 52 + 1.5$$

Mode = 53.5 As Required.

Example:- Calculate Mode from the following Data.

Weight (Kg)	f
40 – 42	10
43 – 45	12
46 – 48	30
49 – 51	6
52 – 54	2

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Solution :- $\frac{43-42}{2} \Rightarrow \frac{1}{2} = .5$

Now,

weight (Kg)	f	
39.5 – 42.5	10	
42.5 – 45.5	12	→ f_1
45.5 – 48.5	30	→ f_m
48.5 – 51.5	6	→ f_2
51.5 – 54.5	2	

(45.5 → l)

$l = 45.5, \; f_1 = 12, \; f_2 = 6$

$f_m = 30$

$h = 48.5 - 45.5 \Rightarrow 3$

$$\text{Mode} = l + \frac{(f_m - f_1)}{(f_m - f_1) + (f_m - f_2)} \times h$$

$$= 45.5 + \frac{(30-12)}{(30-12)+(30-6)} \times 3$$

$$= 45.5 + \frac{18}{18+24} \times 3$$

$$= 45.5 + 0.6923$$

$$= 46.79$$ As Required.

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"5- Median"

The median of set of values arranged in Ascending OR Decending order of magnitude is defined as middle value if the number of the values is odd & mean of two middle values if the number of values is even.

e.g,

5, 6, 7, 8, 9

Median = 7

5, 6, 7, 8, 9, 10

Median $= \frac{7+8}{2} = \frac{15}{2} = 7.5$

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For ungrouped data:-

Median $= \left(\frac{n+1}{2}\right)$-th term

For Grouped Data:-

$$\text{Median} = l + \frac{h}{f}\left(\frac{n}{2} - c\right)$$

where;

l = lower class boundry of Median class.

h = size of the class interval of the median class.

f = frequency of median class

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n = Σf

c = Cumulative frequency of the class preceeding the median class.

Example:- Find Median of the following data.

Weights (Kg)	frequency
56 – 58	25
58 – 60	40
60 – 62	250
62 – 64	130
64 – 66	60
66 – 68	20

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Solution:-

Weights (Kg)	Frequency	Cumulative frequency
56 – 58	25	25
58 – 60	40	65 → C
60 – 62	250	315
62 – 64	130	445
64 – 66	60	505
66 – 68	20	525

(60 → l, 250 → f)

$n = \Sigma f = 525$

$h = 62 - 60$

$= 2$

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Median $= \tilde{x} = l + \frac{h}{f}\left(\frac{n}{2} - c\right)$

$\tilde{x} = 60 + \frac{2}{250}\left(\frac{525}{2} - 65\right)$

$\tilde{x} = 61.58$ AS Required.

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* "QUARTILES"

There are three Quartiles denoted by Q_1, Q_2 & Q_3 & is given by

$Q_1 = \left(\frac{n+1}{4}\right)$ th item

$Q_2 = 2\left(\frac{n+1}{4}\right)$ th item $= \left(\frac{n+1}{2}\right)$ th term = Median

$Q_3 = 3\left(\frac{n+1}{4}\right)$ th item

Quartiles divide the data into 4 Equal Parts.

For Grouped Data:-

$Q_1 = l + \frac{h}{f}\left(\frac{n}{4} - c\right)$

$Q_2 = l + \frac{h}{f}\left(\frac{2n}{4} - c\right) = l + \frac{h}{f}\left(\frac{n}{2} - c\right) =$ Median.

$Q_3 = l + \frac{h}{f}\left(\frac{3n}{4} - c\right)$

"DECILES"

Deciles divide data into 10 Equal Parts, denoted by $D_1, D_2, \ldots\ldots, D_9$ & is given by

$$D_1 = \left(\frac{n+1}{10}\right) \text{th item}$$

$$D_2 = 2\left(\frac{n+1}{10}\right) \text{th item}$$

$$D_3 = 3\left(\frac{n+1}{10}\right) \text{th item}$$

$$\vdots$$

$$D_5 = 5\left(\frac{n+1}{10}\right) \text{th item} = \left(\frac{n+1}{2}\right) \text{th item} = \tilde{X}$$

$$\vdots$$

$$D_9 = 9\left(\frac{n+1}{10}\right) \text{th item}$$

For Grouped Data:-

$$D_1 = l + \frac{h}{f}\left(\frac{n}{10} - c\right)$$

$$D_2 = l + \frac{h}{f}\left(\frac{2n}{10} - c\right)$$

$$\vdots$$

$$D_5 = l + \frac{h}{f}\left(\frac{5n}{10} - c\right) = l + \frac{h}{f}\left(\frac{n}{2} - c\right) = \tilde{X}$$

$$\vdots$$

$$D_9 = l + \frac{h}{f}\left(\frac{9n}{10} - c\right)$$

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PERCENTILES

There are 99 Percentiles denoted by $P_1, P_2, \ldots, P_{99}$. Percentiles divide the data into 100 equal Parts.

For Ungrouped Data:-

$P_1 = \left(\frac{n+1}{100}\right)$ th. item

$P_2 = 2\left(\frac{n+1}{100}\right)$ th item

$\vdots$

$P_{50} = 50\left(\frac{n+1}{100}\right)$ th item $= \left(\frac{n+1}{2}\right)$ th item $= \tilde{X}$

$\vdots$

$P_{99} = 99\left(\frac{n+1}{100}\right)$ th item

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For Grouped Data:-

$P_1 = l + \frac{h}{f}\left(\frac{n}{100} - c\right)$ th item

$P_2 = l + \frac{h}{f}\left(\frac{2n}{100} - c\right)$ th item

$\vdots$

$P_{50} = l + \frac{h}{f}\left(\frac{50n}{100} - c\right)$ th item $= l + \frac{h}{f}\left(\frac{n}{2} - c\right) = \tilde{X}$

$\vdots$

$P_{99} = l + \frac{h}{f}\left(\frac{99n}{100} - c\right)$ th item.

Example :- Find Q_2, Q_3, D_6 & P_{27} from the following data,

X	f
0	1
1	9
2	26
3	59
4	72
5	52
6	29
7	7
8	1

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Solution :-

X	f	C.F
0	1	1
1	9	10
2	26	36
3	59	95
4	72	167
5	52	219
6	29	248
7	7	255
8	1	256

$n = \Sigma f = 256$

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$Q_1 = \left(\frac{n+1}{4}\right)$ th item

$Q_1 = \left(\frac{256+1}{4}\right)$ th item

$= 128.5$ th item

$Q_1 =$ 4th item.

$Q_3 = 3\left(\frac{n+1}{4}\right)$

$= 3\left(\frac{256+1}{4}\right)$

$= 192.75$ th item

$Q_3 =$ 5th item

$D_6 = 6\left(\frac{n+1}{10}\right)$ th item

$= 6\left(\frac{256+1}{10}\right)$ th item

$= 154.2$ th item

$D_6 =$ 4th item.

$P_{27} = 27\left(\frac{n+1}{100}\right)$ th item

$= 27\left(\frac{256+1}{100}\right) = (69.39)$th item

$P_{27} =$ 3rd item.

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MEASURES OF DISPERSION

Dispersion:- By Dispersion we mean that the extent to which the values are spread out from the Average.

Let the following distributions

I - 67, 67, 67, 67, 67, 67, 67 $\bar{X} = 67$

II - 66, 66, 66, 67, 67, 68, 68, 68 $\bar{X} = 67$

III - 52, 53, 61, 67, 71, 72, 78, 82 $\bar{X} = 67$

iv - 43, 44, 50, 55, 66, 90, 91, 97 $\bar{X} = 67$

There are five method of dispersion which are given by,

1) The Range
2) The Quartile Deviation
3) The Mean Deviation
4) Variance
5) Standard Deviation

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1) The Range:-

For ungrouped data

$$Range = X_m - X_o$$

e.g.

65, 66, 100, 23, 40, 20

$X_m = 100,\quad X_o = 20$

$$Range = X_m - X_o$$
$$= 100 - 20 = 80$$

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For Grouped Data:-

Range = $X_m - X_o$

But here

X_m = Upper class boundry of Largest class

X_o = Lower class boundry of smallest class.

e.g,

Classes	f
X_o 10 – 20	3
20 – 30	7
30 – 40	8
40 – 50	10
50 – 60 → X_m	2

So,

$$\text{Range} = X_m - X_o$$

$$= 60 - 10$$

$$= 50$$

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2) The Quartile Deviation:-

$$Q.D = \frac{Q_3 - Q_1}{2}$$

For ungrouped data

$$Q_1 = \frac{n+1}{4}$$

$$Q_3 = 3\left(\frac{n+1}{4}\right)$$

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For Grouped data :-

$$Q_1 = l + \frac{h}{f}\left(\frac{n}{4} - c\right)$$

$$Q_3 = l + \frac{h}{f}\left(\frac{3n}{4} - c\right)$$

*3) The Mean Deviation:-

$$M.D = \frac{\sum |x - \bar{x}|}{n}$$

where $\bar{x}$ = Mean
This formula is used for ungrouped data.

e.g, 65, 66, 100, 23, 40, 20

$$\bar{x} = \frac{\sum x}{n} = \frac{314}{6} = 52.3$$

x	$\lvert x - \bar{x} \rvert$
65	12.6
66	13.7
100	47.7
23	29.3
40	12.3
20	32.3

$$\sum |x - \bar{x}| = 147.9$$

$$\therefore M.D = \frac{147.9}{6} = 24.6$$

Required.

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For Grouped Data

$$M.D = \frac{\Sigma f\,|x-\bar{x}|}{\Sigma f}$$

e.g.

Weights (Kg)	f
56 – 58	25
58 – 60	40
60 – 62	250
62 – 64	130
64 – 66	60
66 – 68	20

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$$\therefore \bar{x} = \frac{\Sigma fx}{\Sigma f}$$

Weights (Kg)	f	x	fx
56 – 58	25	57	1425
58 – 60	40	59	2360
60 – 62	250	61	15250
62 – 64	130	63	8190
64 – 66	60	65	3900
66 – 68	20	67	1340
	$\Sigma f = 525$		$\Sigma fx = 32465$

$$\bar{x} = \frac{32465}{525} = 61.83$$

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$$M.D = \frac{\Sigma f|x-\bar{x}|}{\Sigma f}$$

x	f	$\lvert x-\bar{x}\rvert$	$f\lvert x-\bar{x}\rvert$
57	25	4.83	120.75
59	40	2.83	113.2
61	250	.83	207.5
63	130	2.83	367.9
65	60	4.83	289.8
67	20	6.83	136.6
	$\Sigma f = 525$	Σ	$\Sigma f\lvert x-\bar{x}\rvert = 1235.75$

$$M.D = \frac{1235.75}{525}$$

$$M.D = 2.35$$

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Question:- 7, 4, 10, 9, 15, 12, 7, 9, 7

Calculate Mean Deviation from Mean
Calculate Mean Deviation from Median
Calculate Mean Deviation from Mode

Remark*:-

$$*\ M.D = \frac{\Sigma|x - AVE|}{n}$$

It is General formula for M.D for ungrouped Data.

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for ungrouped data

$$M.D = \frac{\Sigma f|X - AVE|}{\Sigma f}$$ *

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Solution:- Mean $= \frac{\Sigma X}{n}$

$= \frac{80}{9}$

$= 8.89$

Median = 4, 7, 7, 7, 9, 9, 10, 12, 15 = 9

Mode = 7

X	|X – Mean|	|X – Median|	|X – Mode|
4	4.89	5	3
7	1.89	2	0
7	1.89	2	0
7	1.89	2	0
9	0.11	0	2
9	0.11	0	2
10	1.11	1	3
12	3.11	3	5
15	6.11	6	8
	$\Sigma\|X-Mean\| = 21.11$	$\Sigma\|X-Median\| = 21$	$\Sigma\|X-Mode\| = 23$

Also, n = 9

i) Mean Deviation from Mean

$$M.D = \frac{21.11}{9} = 2.35$$

ii) Mean Deviation from Median

$$M.D = \frac{21}{9} = 2.33$$

iii) Mean Deviation from Mode

$$M.D = \frac{23}{9} = 2.55$$

Required.

Question:- 1040, 1080, 1120, 1240, 1260, 1320, 1342, 1360, 1440, 1470, 1600, 1680, 1720, 1730, 1750, 1755, 1785, 1880, 1885, 1960.
Find Range & Quartile Deviation.

Solution:- Range = $X_m - X_o$

$X_m = 1960$

$X_o = 1040$

$R = X_m - X_o$

$R = 1960 - 1040$

$R = 920$

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Q_1 = Value of $\left(\frac{n+1}{4}\right)$th term

= value of $\left(\frac{20+1}{4}\right)$th term

= value of (5.25)th item

Q_1 = 5th item + 0.25 (6th item − 5th item)

= 1260 + 0.25(1320 − 1260)

= 1260 + 15

Q_1 = 1275

Q_3 = Value of $3\left(\frac{n+1}{4}\right)$th term

Q_3 = value of $3\left(\frac{20+1}{4}\right)$th term

Q_3 = value of (15.75)th term

= 15th item + .75(16th term − 15th term)

= 1750 + .75(1755 − 1750)

= 1750 + 3.75

Q_3 = 1753.75

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$$Q.D = \frac{Q_3 - Q_1}{2}$$

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$$= \frac{1753.75 - 1275}{2}$$

$$Q.D = 239.375 \quad \text{Required.}$$

4- Variance

The Variance of the observation is defined as "The Mean of the square of Deviations of observations taken from the Mean of the observation. Variance is denoted by S^2.

$$S^2 = \frac{\Sigma(X - \bar{X})^2}{n} \quad \text{(By Def)}$$

Variance by short cut Method OR Direct Method.

$$S^2 = \frac{\Sigma(X - \bar{X})^2}{n}$$

$$S^2 = \frac{\Sigma\left[X^2 + \bar{X}^2 - 2X\bar{X}\right]}{n}$$

$$S^2 = \frac{\Sigma X^2}{n} + \frac{\Sigma \bar{X}^2}{n} - 2\bar{X}\frac{\Sigma X}{n}$$

$$S^2 = \frac{\Sigma X^2}{n} + \frac{n\bar{X}^2}{n} - 2\bar{X}\bar{X}$$

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$+ \bar{X}^2 - 2\bar{X}^2$

$= \frac{\Sigma x^2}{n} - \bar{X}^2$

$$S^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$$

For Grouped Data,

$$S^2 = \frac{\Sigma f (x - \bar{x})^2}{\Sigma f}$$

Question:- 6, 18, 20, 27, 34

Find Variance

i) by Def

ii) by Direct Method.

Solution

$$\bar{x} = \frac{\Sigma x}{n}$$

$$\bar{x} = \frac{105}{5} = 21$$

x	$(x-\bar{x})^2$
6	225
18	9
20	1
27	36
34	169

$\Sigma (x - \bar{x})^2 = 440$

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$n = 5$

So,
$$S^2 = \frac{\Sigma (X-\bar{X})^2}{n}$$
$$= \frac{440}{5}$$
$$S^2 = 88 .$$ As Required.

ii)

X	X^2
6	36
18	324
20	400
27	729
34	1156
$\Sigma X = 105$	$\Sigma X^2 = 2645$

$n = 5$

So,
$$S^2 = \frac{\Sigma X^2}{n} - \left(\frac{\Sigma X}{n}\right)^2$$
$$= \frac{2645}{5} - \left(\frac{105}{5}\right)^2$$
$$= 529 - 441$$
$$S^2 = 88$$

Required.

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5- Standard Deviation

For Un Grouped Data

$$S = \sqrt{\frac{\Sigma (X-\bar{X})^2}{n}}$$

For Grouped Data,

$$S = \sqrt{\frac{\Sigma f(X-\bar{X})^2}{\Sigma f}}$$

e.g, 6, 18, 20, 27, 34

$\bar{X} = 21$

$S^2 = 88$

$S = \sqrt{88}$

$S = 9.38$ As Required.

Also,

$$S = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2}$$

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Q2) The Mean & G.M of three numbers are 7 & 4 respectively. Find all three numbers if mean of first two number is 10.

Solution:- Let A, B, C be three required number then by given condition,

$$A.M = \frac{A+B+C}{3}$$

$$7 = \frac{A+B+C}{3}$$

$$A+B+C = 21 \rightarrow (1)$$

$$G.M = (A \cdot B \cdot C)^{1/3}$$

$$4 = (A \cdot B \cdot C)^{1/3}$$

$$ABC = (4)^3$$

$$ABC = 64 \rightarrow (2)$$

Also given

$$A.M = \frac{A+B}{2}$$

$$10 = \frac{A+B}{2}$$

$$A+B = 20 \rightarrow (3)$$

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Put $A+B=20$ in eq (1)

$$20+C=21$$

$$C=21-20$$

$$C=1$$

Put in (2)

$$AB(1)=64$$

$$AB=64$$

From eq (3)

$$B=20-A$$

So,

$$A(20-A)=64$$

$$-A^2+20A=64$$

$$A^2-20A+64=0$$

$$A^2-16A-4A+64=0$$

$$A(A-16)-4(A-16)=0$$

$$(A-16)(A-4)=0$$

Either $A=4$

OR $A=16$

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So,
$$B = 20 - A$$
$$B = 20 - 4$$
$$B = 16$$
Also
$$B = 20 - A$$
$$B = 20 - 16$$
$$B = 4$$

∴ Either $B = 4$ OR $B = 16$

∴

$A = 4$, $B = 16$ & $C = 1$ are Required.

Question 1) Find the two items such that A.M & G.M of two items are 12.5 & 10 respectively.

Solution:- Let A & B are Required.

So, that
$$12.5 = \frac{A+B}{2}$$
$$A + B = 25 \rightarrow (1)$$
$$10 = (AB)^{1/2}$$
$$AB = (10)^2$$
$$AB = 100 \rightarrow (2)$$

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From ①

$A = 25 - B$

So eq ② becomes

$(25 - B) B = 100$

$25B - B^2 = 100$

$B^2 - 25B + 100 = 0$

$B^2 - 20B - 5B + 100 = 0$

$B(B-20) - 5(B-20) = 0$

$(B-20)(B-5) = 0$

∴ Either $B = 20$ OR $B = 5$

∴ $A = 25 - B$

$A = 25 - 5$

$A = 20$

$A = 25 - B$

$A = 25 - 20$

$A = 5$

∴ Either $A = 5$ OR $A = 20$

$A = 20$ & $B = 5$ are Required.

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Question 3) Find Arthimatic Mean of the Variable X, from the following,

u	-4	-3	-2	-1	0	1	2	3
f	12	15	28	48	62	23	18	14

where $u = \frac{X - 136.5}{2}$

Solution:-

Here, $X = 2u + 136.5$

$X = 2u + 136.5$	f	fX
128.5	12	1542
130.5	15	1957.5
132.5	28	3710
134.5	48	6456
136.5	62	8436
138.5	23	3185.5
140.5	18	2529
142.5	14	1995
	$\Sigma f = 220$	$\Sigma fX = 29811$

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$\therefore \quad A.M = \frac{\Sigma fX}{\Sigma f}$

$= \frac{29811}{220}$

$A.M = 135.5$ Required.

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Question 4) Calculate Median from the following Data,

Classes	40–42	43–45	46–48	49–51	52–54
f	16	12	30	6	2

Solution:- $\frac{43-42}{2} \Rightarrow \frac{1}{2} \Rightarrow .5$

Now

Classes	f	C.F
39.5 – 42.5	16	16
42.5 – 45.5	12	28 → C
45.5 – 48.5	30 → f	58
48.5 – 51.5	6	64
51.5 – 54.5	2	66

l

$n = \Sigma f = 66$

$h = 48.5 - 45.5 = 3$

$$\text{Median} = \tilde{x} = l + \frac{h}{f}\left(\frac{n}{2} - c\right)$$

$$= 45.5 + \frac{3}{30}\left(\frac{66}{2} - 28\right)$$

$$= 45.5 + \frac{1}{10}(33 - 28)$$

$$= 45.5 + 0.5$$

$$\tilde{x} = 46 \quad \text{Required.}$$

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Question 5) The mean Salary Paid to 1000 employes of a Factory was found to be Rs: 180.4 Lateron it was discovered that wages of two employes were wrongly taken as 297 & 165 instead of 197 & 185. Find Correct Mean.

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Sol :- $n = 1000$ $\bar{x} = 180.4$

Wrong values = 297, 165.

Correct Values = 197, 185

Uncorrect :-

$$\text{Uncorrect } \bar{x} = \frac{\text{uncorrect } \Sigma x}{n}$$

$$180.4 = \frac{\text{uncorrect } \Sigma x}{1000}$$

$$180400 = \text{uncorrect } \Sigma x$$

$$\text{Correct } \Sigma x = \text{Uncorrect } \Sigma x - (\text{Wrong value}) + (\text{Right Value})$$

$$= 180400 - (297+165) + (197+185)$$

$$\text{Correct } \Sigma x = 180320$$

$$\text{Correct } \bar{x} = \frac{\text{Correct } \Sigma x}{n}$$

$$= \frac{180320}{1000}$$

$$= 180.32$$

Required.

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Relative Measures of Dispersion

Co-efficient of Range $= \dfrac{X_m - X_o}{X_m + X_o}$

Co-efficient of Q.D $= \dfrac{Q_3 - Q_1/2}{Q_3 + Q_1/2}$

$= \dfrac{Q_3 - Q_1}{Q_3 + Q_1}$

Co-efficient of M.D $= \dfrac{M.D}{Mean}, \dfrac{M.D}{Median}, \dfrac{M.D}{Mode}$

Co-efficient of Deviation $= \dfrac{S}{\bar{X}}$

Co-efficient of Variation $= \dfrac{S}{\bar{X}} \times 100$

Moments:- The Mean of different Powers of Deviations from Mean.

$$m_1 = \frac{\Sigma (X - \bar{X})}{n}$$

$$m_2 = \frac{\Sigma (X - \bar{X})^2}{n}$$

$$m_3 = \frac{\Sigma (X - \bar{X})^3}{n}$$

$$m_4 = \frac{\Sigma (X - \bar{X})^4}{n}$$

These moments for Ungrouped data.

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For Grouped Data :-

$$m_1 = \frac{\Sigma f(x-\bar{x})}{\Sigma f}$$

$$m_2 = \frac{\Sigma f(x-\bar{x})^2}{\Sigma f}$$

$$m_3 = \frac{\Sigma f(x-\bar{x})^3}{\Sigma f}$$

$$m_4 = \frac{\Sigma f(x-\bar{x})^4}{\Sigma f}$$

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The above Moments about Mean.

Symmetry:- A distribution is said to be symmeteric if

Mean = Median = Mode

e.g.

Group	freq
0 – 10	20
10 – 20	25
20 – 30	40
30 – 40	25
40 – 50	20

Median = Mode = Mean

= 25

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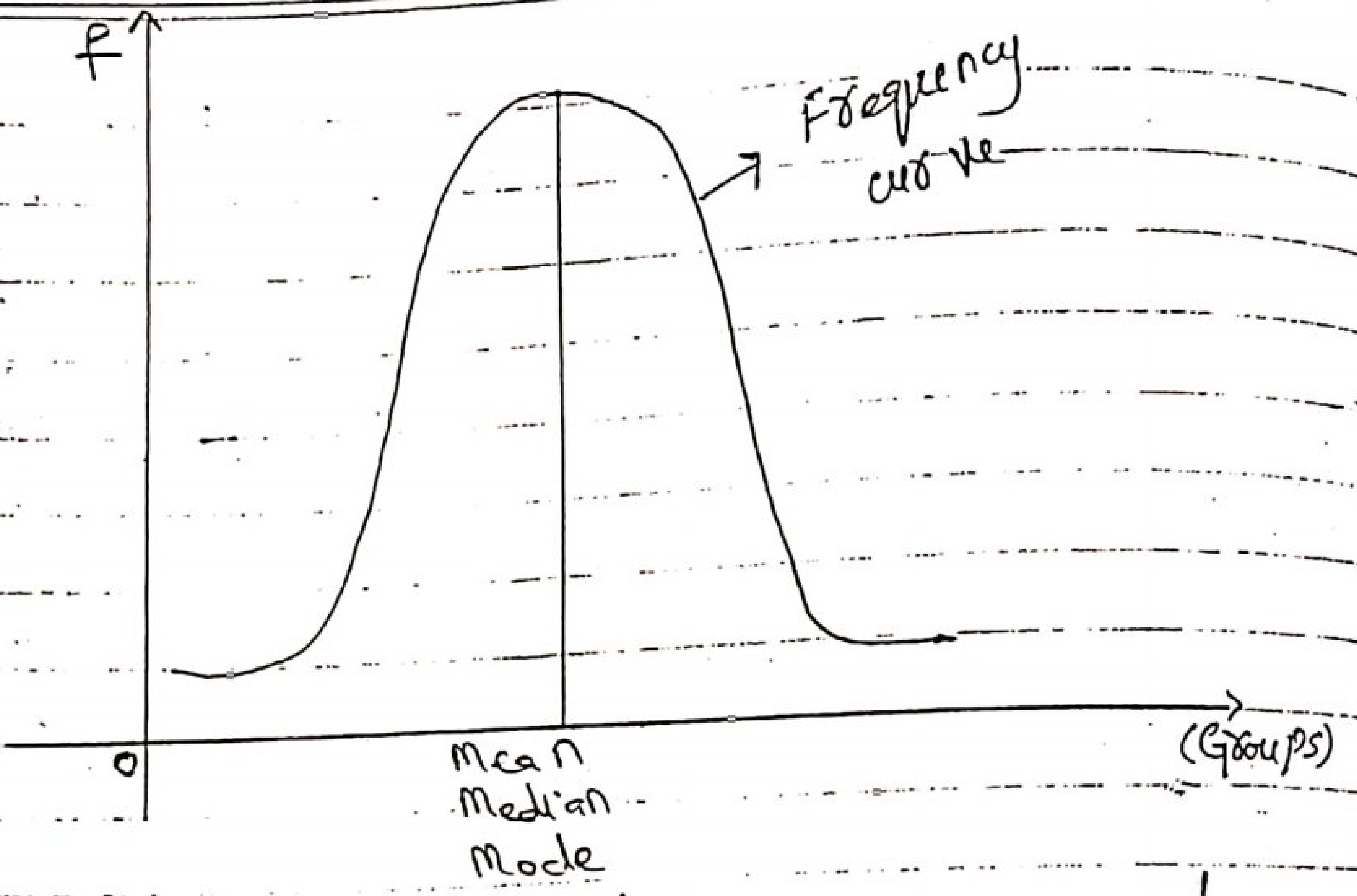
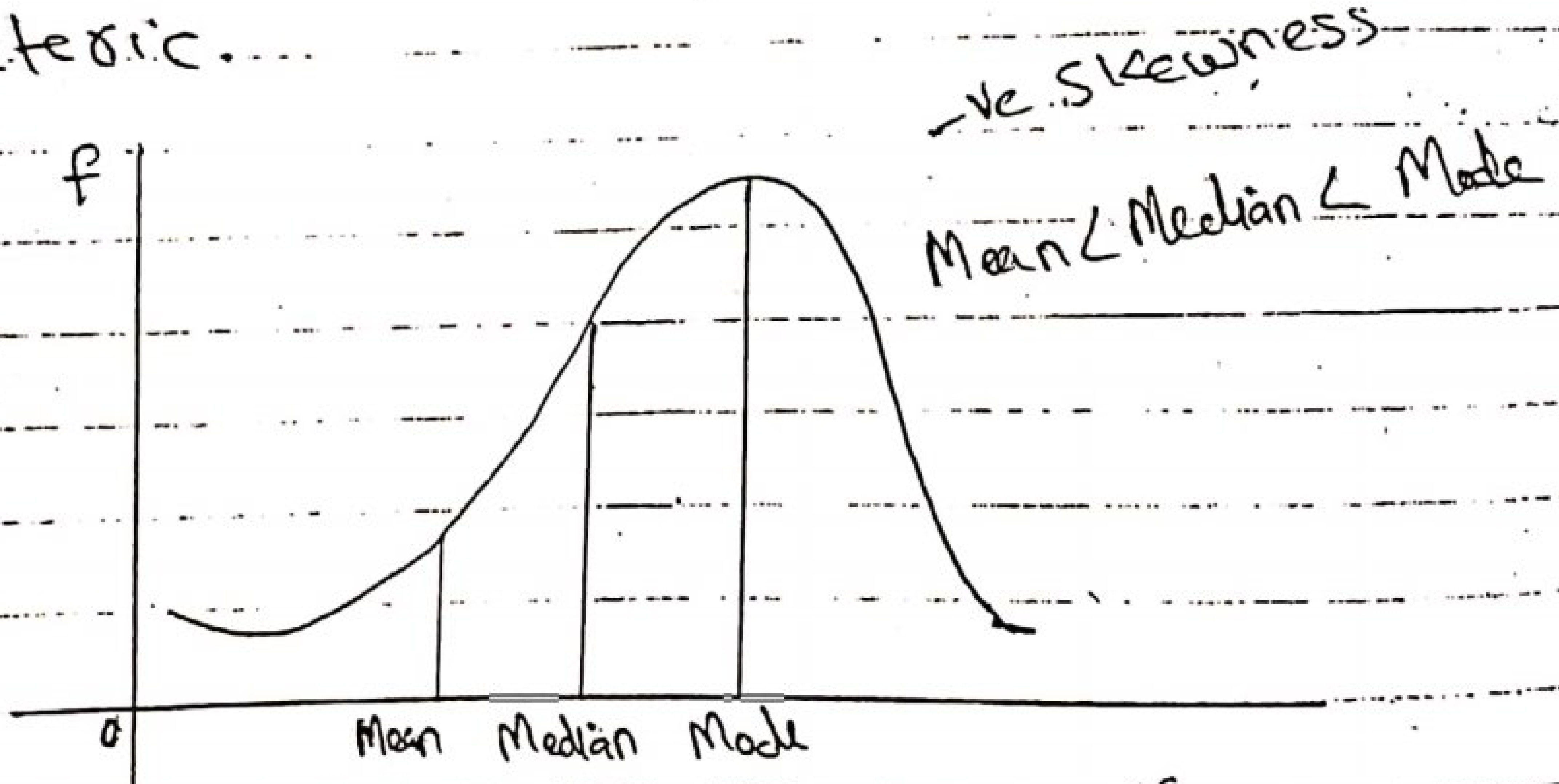
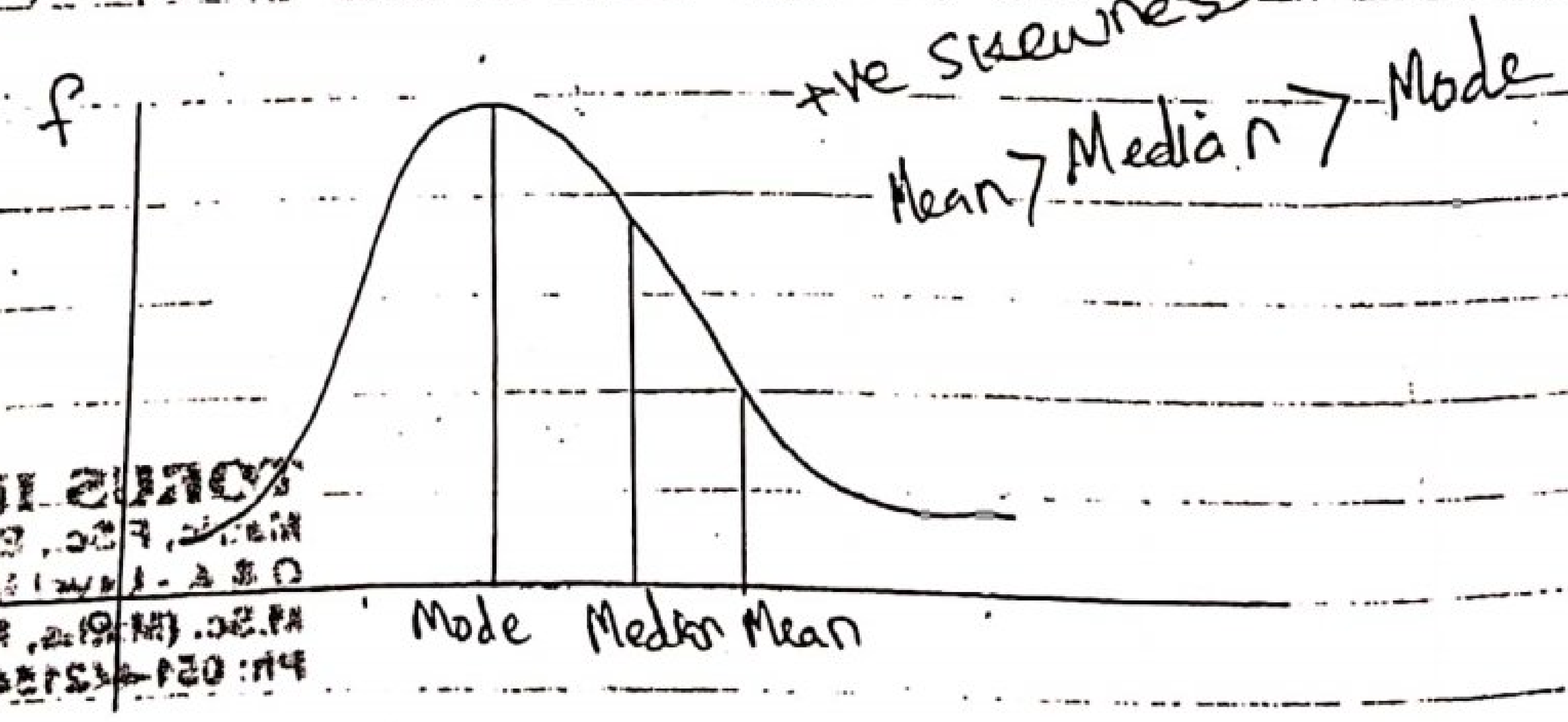


SKEWNESS:- A distribution is said to be skewness if it is not symmeteric.





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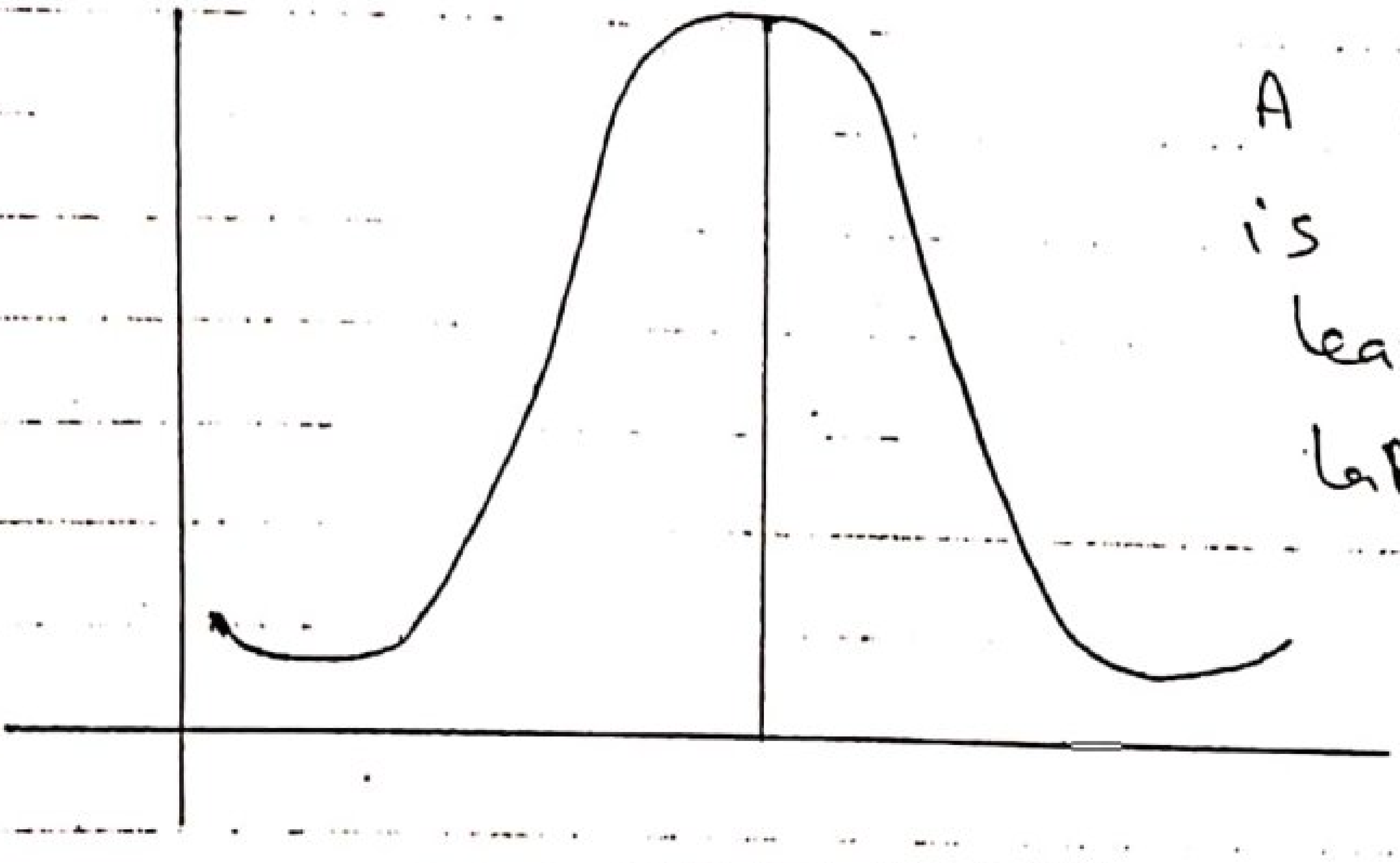
KURTOSIS:- The term Kurtosis is used for Convexity OR Peakedness of the frequency curve.



A curve which is tall & lean is called Lapto Kurtic.

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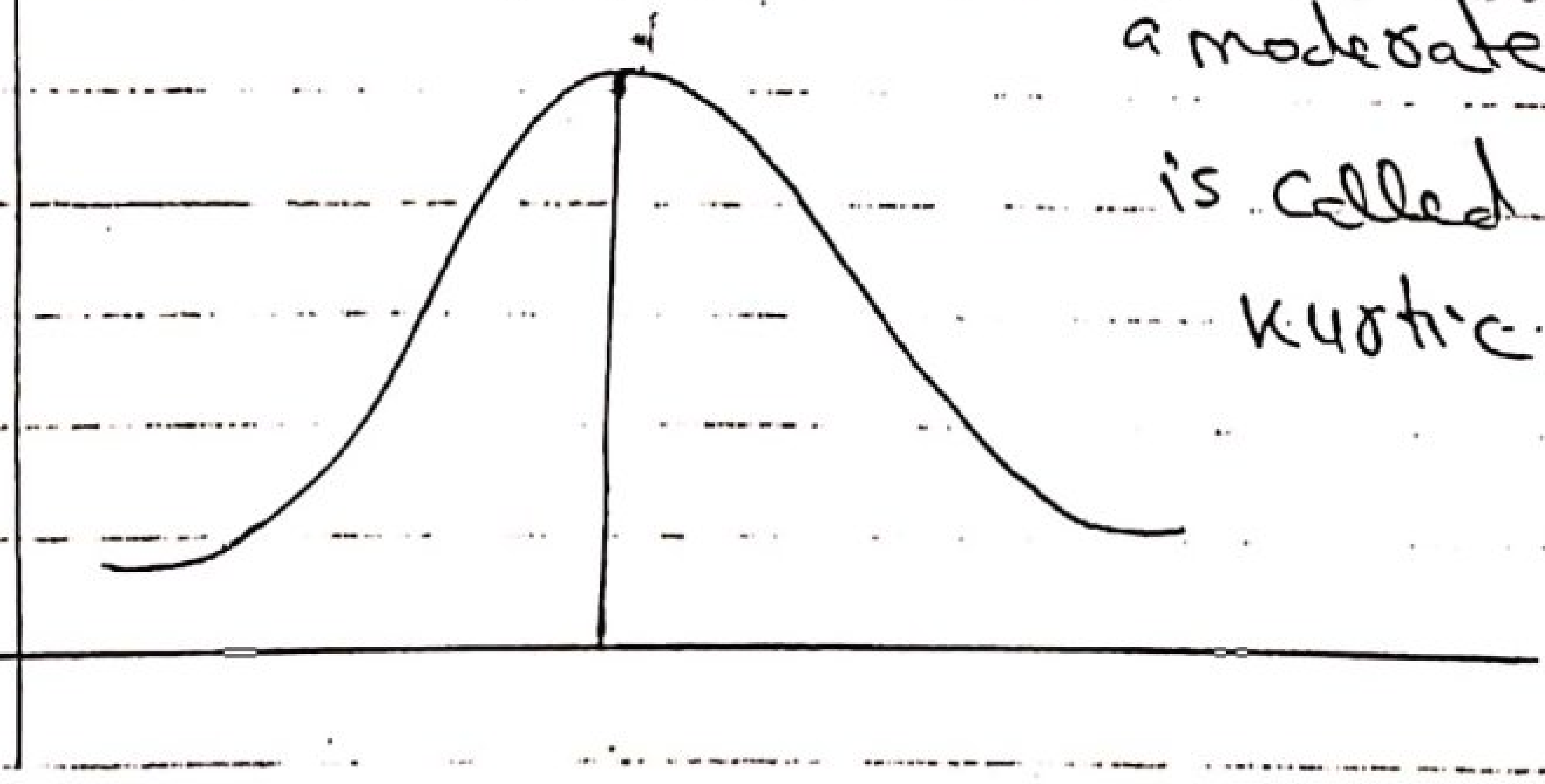
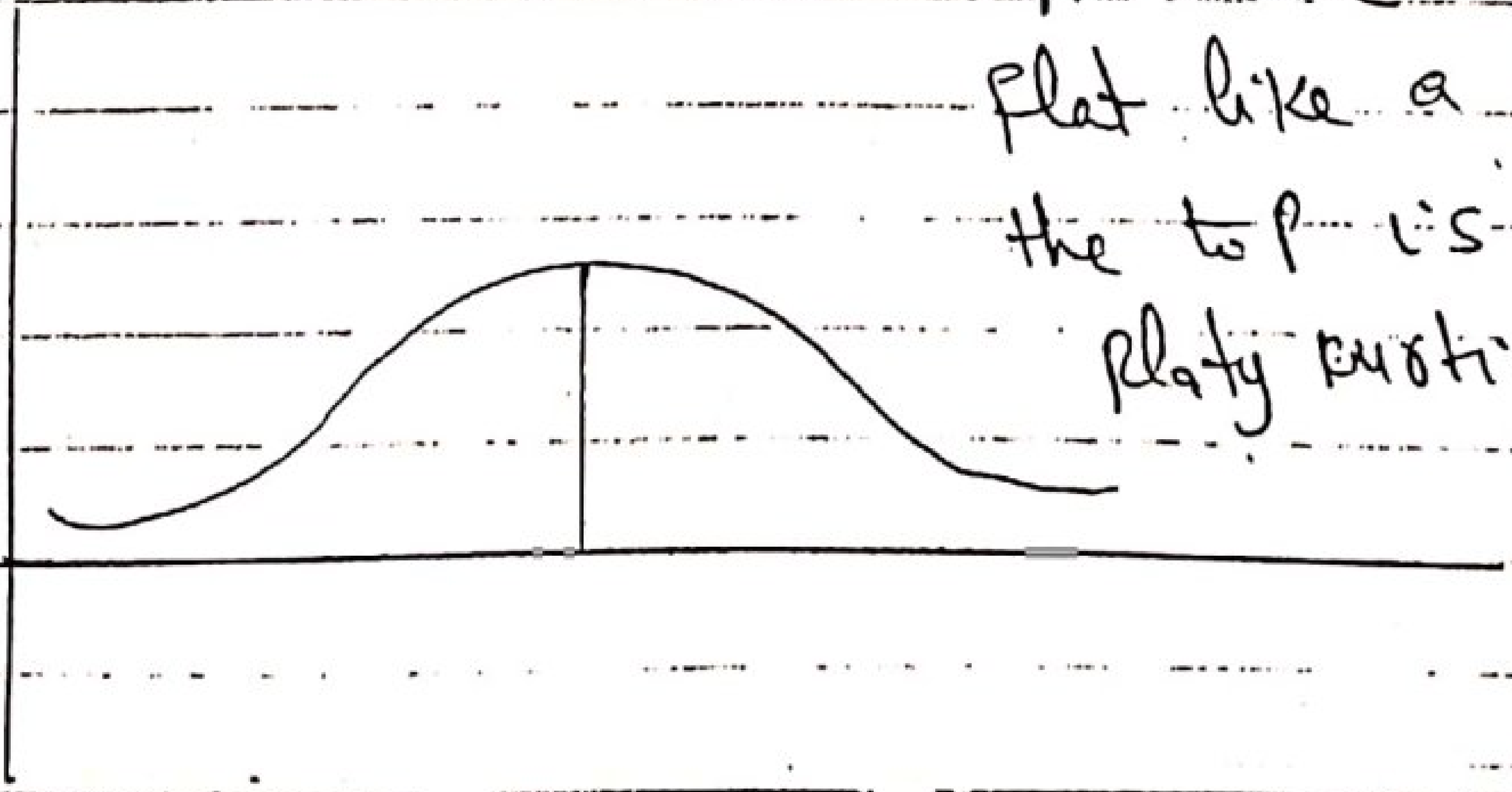


A curve with a moderate Peak is called Meso kurtic (Normal)



A curve which is flat like a Plate at the top is called platy kurtic.

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Moment Ratio:- Based on 1st 4-moments about means, two Ratios have been defined

$$b_1 = \frac{m_3^2}{m_2^3} \text{ [Co-efficient of skewness]}$$

$$b_2 = \frac{m_4}{m_2^2} \text{ [Co-efficient of Kurtosis]}$$

If $b_2 > 3$ then is Lapto kurtic

If $b_2 < 3$ then is Platy kurtic

If $b_2 = 3$ then is Meso kurtic (Normal)

If $b_1 = 0$ then is symmetric

If $b_1 \neq 0$ then is skewness

& skewness is +ve if $m_3 > 0$.

Problem:- Calculate 1st four moments about mean of the following data.

8, 10, 15, 22, 26, 30, 35, 40, 48

Solution:-

$$\bar{X} = \frac{\Sigma X}{n} = \frac{234}{9} = 26$$

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X	$(X-\bar{X})$	$(X-\bar{X})^2$	$(X-\bar{X})^3$	$(X-\bar{X})^4$
8	-18	324	-5832	104976
10	-16	256	-4096	65536
15	-11	121	-1331	14641
22	-4	16	-64	256
26	0	0	0	0
30	4	16	64	256
35	9	81	729	6561
40	14	196	2744	38416
48	22	484	10648	234256
	$\Sigma(X-\bar{X}) = 0$	$\Sigma(X-\bar{X})^2 = 1494$	$\Sigma(X-\bar{X})^3 = 2862$	$\Sigma(X-\bar{X})^4 = 464898$

$$m_1 = \frac{\Sigma(X-\bar{X})}{n} = \frac{0}{9} = 0$$

$$m_2 = \frac{\Sigma(X-\bar{X})^2}{n} = \frac{1494}{9} = 166$$

$$m_3 = \frac{\Sigma(X-\bar{X})^3}{n} = \frac{2862}{9} = 318$$

$$m_4 = \frac{\Sigma(X-\bar{X})^4}{n} = \frac{464898}{9} = 51655.33$$

Problem 2) Calculate 1st four moments about mean & also calculate b_1, b_2 & interpret them.

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X	f
12	4
13	11
14	32
15	21
16	15
17	8
18	5
20	4

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Solution:-

X	f	fX	$(X-\bar{X})$	$f(X-\bar{X})^1$	$f(X-\bar{X})^2$	$f(X-\bar{X})^3$	$f(X-\bar{X})$
12	4	48	-3	-12	36	-108	324
13	11	143	-2	-22	44	-88	176
14	32	448	-1	-32	32	-32	32
15	21	315	0	0	0	0	0
16	15	240	1	15	15	15	15
17	8	136	2	16	32	64	128
18	5	90	3	15	45	135	405
20	4	80	5	20	100	500	2500
		$\Sigma fX = 1500$	$\Sigma(X-\bar{X}) = 5$	$\Sigma f(X-\bar{X}) = 0$	$\Sigma f(X-\bar{X})^2 = 304$	$\Sigma f(X-\bar{X})^3 = 486$	$\Sigma f(X-\bar{X}) = 3580$

$$\bar{X} = \frac{\Sigma fX}{\Sigma f}$$

$$\bar{X} = \frac{1500}{100} = 15$$

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$$m_1 = \frac{\sum f(x-\bar{x})}{n\sum f} = \frac{0}{100} = 0$$

$$m_2 = \frac{\sum f(x-\bar{x})^2}{\sum f} = \frac{304}{100} = 3.04$$

$$m_3 = \frac{\sum f(x-\bar{x})^3}{\sum f} = \frac{486}{100} = 4.86$$

$$m_4 = \frac{\sum f(x-\bar{x})^4}{\sum f} = \frac{3580}{100} = 35.80$$

Now

$$b_1 = m_3^2 / m_2^3 = \frac{(4.86)^2}{(3.04)^3} = 0.84$$

$\Rightarrow$ $b_1 \neq 0$

So, Distribution is +vely skewed

$\because$ m_3 is +ve.

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Now $b_2 = m_4 / m_2^2$

$$b_2 = \frac{35.8}{(3.04)^2} = 3.87$$

$\Rightarrow$ $b_2 > 3$

$\Rightarrow$ Distribution is leptokurtic.

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KARL Person suggested two measures of skewness.

i) Co-efficient of skewness

$$= \frac{\text{Mean} - \text{Mode}}{S}$$

ii) Co-efficient of skewness

$$= \frac{3(\text{Mean} - \text{Median})}{S}$$

where

S is standard Deviation.

$$S = \sqrt{\frac{\Sigma(X - \bar{X})^2}{n}}$$

OR

$$S = \sqrt{\frac{\Sigma X^2}{n} - \left(\frac{\Sigma X}{n}\right)^2}$$

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Remark:- If Co-efficient of skewness = +ve then distribution is +ve skewness.
If Co-efficient of skewness = −ve then distribution is −ve skewness.

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If Co-efficient of skewness = 0 then distribution is symmeteric.

Bowely's Co-efficient of Skewness

$$= \frac{(Q_3 - Median) - (Median - Q_1)}{(Q_3 - Median) + (Median - Q_1)}$$

OR

$$= \frac{Q_3 + Q_1 - 2Median}{Q_3 - Q_1}$$

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Remark:- we have 4 methods for check the skewness.

Raw Moments

Raw Moments about orbitary origin

$D = X - A$

$m_1' = \frac{\Sigma fD}{\Sigma f}$, $m_3' = \frac{\Sigma fD^3}{\Sigma f}$

$m_2' = \frac{\Sigma fD^2}{\Sigma f}$, $m_4' = \frac{\Sigma fD^4}{\Sigma f}$

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Relation b/w Raw Moments & Moments about Mean

$$m_1 = 0$$

$$m_2 = m_2' - (m_1')^2$$

$$m_3 = m_3' - 3m_1' m_2' + 2m_1'^3$$

$$m_4 = m_4' - 4m_1' m_3' + 6(m_1')^2(m_2') - 3m_1'^4$$

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CHAPTER # 3

"PROBABILITY"

Experiment:- The term Experiment means a well defined action OR a process of obtaining an observation. A single Performence of an experiment is called a trial. The result obtained from an Experiment is called an outcome.

Random Experiment:- An Experiment is called Random Experiment if it Produces different results even if it is repeated for large number of time.

SAMPLE SPACE:- It is set of all possible outcomes of a Random Experiment.
It is denoted by S.
e.g.,

① Tassing a Coin

$S = \{H, T\}$

② Throwing a dice

$S = \{1, 2, 3, 4, 5, 6\}$

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③ Tossing of two Coins

$S = \{HH, HT, TH, TT\}$

Event:- An Event is an indivitual outcome. In set termonology any subset of sample space is called an Event.

e.g.,

$A = \{H\}$

$A \subseteq S = \{H, T\}$

$B = \{T\}$

$B \subseteq S \subseteq \{H, T\}$

$A = \{2, 4, 6\}$, $B = \{1, 3, 5\}$

$A \subseteq S = \{1, 2, 3, 4, 5, 6\}$

$B \subseteq S = \{1, 2, 3, 4, 5, 6\}$

are examples of Event.

Simple Event:- An Event having only one simple point is called simple Event.

e.g., $A = \{H\}$
is simple Event.

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Compound Event:- An Event having more than one simple point is called compound Event.

e.g,

$A = \{2, 4, 6\}$

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Impossible Event:- An Element having no element is called impossible event.

Sure Event:- If

$A = S$ then

A is called Sure Event

Equally like Events:- Two Events A & B are said to be Equally likes Event if they have Equal chances of occures

e.g, $S = \{1, 2, 3, 4, 5, 6\}$

$A = \{1\}$, $B = \{5\}$

Mutually Exclusive Events:- Two Events A & B are said to be mutually Exclusive if they can not occure togeather.

i.e, $A \cap B = \varphi$

$S = \{1, 2, 3, 4, 5, 6\}$

$A = \{1, 3, 5\}$, $B = \{2, 4, 6\}$

then $A \cap B = \varphi$.

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Dependent Events :- Two events A & B are said to be Dependent if the occurence of one event depend upon the occurence OR non-occurence of the other.

e.g,

$S = \{1,2,3,4,5,6\}$

$A = \{2,4,6\}$

$B = \{4\}$

Mutually Exhastive Events :-

When a sample space is Partioned into some mutually Exclusive events such that their union is sample space is itself.

e.g,

$S = \{1,2,3,4,5,6\}$

$A = \{1,2\}$

$B = \{3,4,5\}$

$C = \{6\}$

then $A \cup B \cup C = S$

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Probability:- let there are "n" Possible outcomes of a random experiment & "m" are favourable to event A then

$$P(A) = \frac{m}{n}$$

OR

It is Ratio of number of sample points of A to number of sample points belonging to S.

$$P(A) = \frac{n(A)}{n(S)}$$

Mathematical Def of Probability:-

Probability is non-negative set function defined on a sample space S. If A is event of S then

i) $0 \leq P(A) \leq 1$

ii) $P(\varphi) = 0$

Problem 1) Experiment: A Coin is tossed Find the probability

i) Head Appears

ii) Tail Appears

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Solution :- $S = \{H, T\}$

i) $A = \{H\}$

$$P(A) = \frac{n(A)}{n(S)} = \frac{1}{2} = 0.5$$

ii) $B = \{T\}$

$$P(B) = \frac{n(B)}{n(S)} = \frac{1}{2} = 0.5$$

Problem 2) Experiment: Two Coins are tossed. Find the probability.

i) Exactly two head Appears
ii) Atleast one tail Appears
iii) Three tails Appears.

Solution :- $S = \{HH, HT, TH, TT\}$

i) $A = \{HH\} \Rightarrow P(A) = \frac{n(A)}{n(S)}$

$$P(A) = \frac{1}{4} = 0.25$$

ii) $B = \{HT, TH, TT\} \Rightarrow P(B) = \frac{n(B)}{n(S)}$

$$P(B) = \frac{3}{5} = 0.75$$

iii) $C = \{\varphi\}$

$$P(C) = \frac{0}{5} = 0$$

Required.

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Problem:- A fair dice is thrown Find the probability.

i) Odd numbers Appears
ii) Even numbers Appears
iii) Multiple of three appears

Solution:- $S = \{1, 2, 3, 4, 5, 6\}$

i) $A = \{1, 3, 5\}$

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{6}$$

$$P(A) = 0.50$$

ii) $B = \{2, 4, 6\}$

$$P(B) = \frac{n(B)}{n(S)} = \frac{3}{6}$$

$$P(B) = 0.50$$

iii) $C = \{3, 6\}$

$$P(C) = \frac{n(C)}{n(S)} = \frac{2}{6}$$

$$P(C) = 0.33$$

Required.

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Problem:- Two dice Thrown. Find the probability.

i) Sum 8 appears in dice
ii) Sum 13 appears in dice

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	(8)
3	4	5	6	7	(8)	9
4	5	6	7	(8)	9	10
5	6	7	(8)	9	10	11
6	7	(8)	9	10	11	12

$n(S) = 36$, $A = \{8, 8, 8, 8, 8\}$, $B = \{\}$

i) $P(A) = \frac{n(A)}{n(S)} = \frac{5}{36} = 0.13$

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ii) $P(B) = \frac{n(B)}{n(S)}$

$= \frac{0}{36}$

$= 0$

Required.

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"Use of Combination in Probability"

Example 1) A bag Contain 4 Red, 5 White & 6 Green balls.

a) One ball is selected.

Find the Probability

1) It is Red 2) Not Red
3) Red OR white.

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Solution:-

Red	white	Green	Total
4	5	6	= 15

i) $P(\text{Red Ball}) = \frac{4}{15} = 0.26$

ii) $P(\text{Not Red Ball}) = {}^{11}/_{15} = 0.73$

iii) $P(\text{Red OR white}) = {}^{9}/_{15} = 0.6$

b) Two balls are Selected.
Find the Probability.

i) They are Red

ii) One Red & One Green

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$$n(S) = {}^{15}C_2 = 105$$

i) $P[\text{two Red}] = \dfrac{\binom{4}{2}\binom{5}{0}\binom{6}{0}}{105}$

$$= \frac{6}{105}$$

$$= 0.05$$

ii) $P[\text{one Red \& one Green}] = \dfrac{\binom{4}{1}\binom{6}{1}\binom{5}{0}}{105}$

$$= \frac{24}{105}$$

$$= 0.22$$

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CHAPTER # 4

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RANDOM VARIABLES

Defination:- A Variable which is determined by numerical Values which we assign to a random Experiment is called Random Variable (r.V).

e.g, If we toss a coin then

r.V, $x = 0, 1$

If we toss two Coins then

r.V, $x = 0, 1, 2$

where x denotes the number of heads.

If we throw a dice then r.V is

$x = 1, 2, 3, 4, 5, 6$

where x denotes the number of spots.

Types of RANDOM VARIABLES

Two Types of RANDOM VARIABLES

1- Discrete Random Variables

2- Continous Random Variables

1) Discrete Random Variable:-

A Variable which assume values in whole number e.g, If we toss two Coins then r.V. $x = 0, 1, 2$ where x denotes the number

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of heads then is called discrete random Variable.

2) **Continous Random Variable:-** A Variable which assumes any fractional Value b/w two specific Points say "a" & "b" is called a Continous random Variable.
e.g, height of Person.
temperature at Place.

Probability Distribution OR Probability Function of A Discrete Random Variable :-

A Table listing all the Values of a Random Variable togeather with the Probabilities is called Probability distribution.

Example 1) Find Probability Distribution of x when one Coin is tossed & "x" denotes the number of heads.

Solution:-

X	P(X) = f(X)
0	1/2
1	1/2

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Example 2) Find, Probability Dist of "x" when two Coins are tossed & x denotes the number of heads.

Solution:-

X	P(X) = f(X)
0	1/4
1	2/4
2	1/4

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$S = \{TT, HT, TH, HH\}$
0 1 1 2

$P(0) = 1/4$
$P(1) = 2/4$
$P(2) = 1/4$

Example 3) Find Probability Dist of x if we throw a dice where x denotes the no of spots.

X	P(X) = f(x)
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

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Remark:- A Probability Dist has the following Properties

i) $f(x_i) \geq 0 \;\forall\; x_i$.

ii) $\sum f(x_i) = 1$

Example 4) From an urn containing 4 Red & 6 white balls. A man draws three Balls at random. If x is a r.v which denotes the number of red balls drawn, what is probability dist of X.

Solution:-

Red	white	Total
4	6	10

$$P[\text{no Red Ball}] = P[X=0] = \frac{\binom{4}{0}\binom{6}{3}}{\binom{10}{3}} = 5/30$$

$$P[\text{one Red Ball}] = P[X=1] = \frac{\binom{4}{1}\binom{6}{2}}{\binom{10}{3}} = 15/30$$

$$P[\text{two Red Ball}] = P[X=2] = \frac{\binom{4}{2}\binom{6}{1}}{\binom{10}{3}} = 9/30$$

$$P[\text{three Red Ball}] = P[X=3] = \frac{\binom{4}{3}\binom{6}{0}}{\binom{10}{3}} = 1/30$$

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X	P(X) = f(X)
0	5/30
1	15/30
2	9/30
3	1/30

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Example 5) Given the discrete Probability dist

$$P(X) = \binom{4}{x} \left(\frac{1}{2}\right)^{x} \left(\frac{1}{2}\right)^{4-x} \quad ; X = 0,1,2,3,4$$

Find Probability Dist.

X	P(X)
0	$P(X) = \binom{4}{0}\left(\frac{1}{2}\right)^{0}\left(\frac{1}{2}\right)^{4-0} = 1/16$
1	$P(X) = \binom{4}{1}\left(\frac{1}{2}\right)^{1}\left(\frac{1}{2}\right)^{4-1} = 4/16$
2	$P(X) = \binom{4}{2}\left(\frac{1}{2}\right)^{2}\left(\frac{1}{2}\right)^{4-2} = 6/16$
3	$P(X) = \binom{4}{3}\left(\frac{1}{2}\right)^{3}\left(\frac{1}{2}\right)^{4-3} = 4/16$
4	$P(X) = \binom{4}{4}\left(\frac{1}{2}\right)^{4}\left(\frac{1}{2}\right)^{4-4} = 1/16$

which is Required Probability Dist.

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Example 6) A Random Variable X has the following Probability Dist.

X	-2	-1	0	1	2	3
$P(x)$	0.1	K	0.2	$2K$	0.3	$3K$

Find the following

i) K

ii) $P(X \leq 2)$

iii) $P(X \geq 2)$

iv) $P(-2 < X < 2)$

v) $P(X \leq 1)$

Sol)

i) As Given Table is Probability Dist

So $\sum P(x_i) = 1 \quad \forall x_i$

$$0.1 + K + 0.2 + 2K + 0.3 + 3K = 1$$

$$0.6 + 6K = 1$$

$$6K = 0.4$$

$$K = 0.4/6$$

$$K = 1/15$$

Now Table takes form,

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$P(X \leq 2) = 1 - P(X > 2) = 1 - (3/15) = 1/5 = 0.2$

X	-2	-1	0	1	2	3
P(X)	0.1	1/15	0.2	2/15	0.3	3/15

ii) $P(X \leq 2) = P[X=-2] + P[X=-1] + P[X=0] + P[X=2] + P[X=1]$

$= 0.1 + 1/15 + 0.2 + 2/15 + 0.3$

$= 0.5 + 0.3 = 0.8$

iii) $P(X \geq 2) = P[X=2] + P[X=3]$

$= 0.3 + 3/15$

$= 0.5$

iv) $P(-2 < X < 2) = P[X=-1] + P[X=0] + P[X=1]$

$= 1/15 + 0.2 + 2/15$

$= 0.4$

v) $P(X \leq 1) = P[X=-2] + P[X=-1] + P[X=0] + P[X=1]$

$= 0.1 + 1/15 + 0.2 + 2/15$

$= 0.5$

Required.

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Probability Density Function (P.d.F)

The Probability Dist of a continous (r.v) is called Probability density function (P.d.f).
It cannot be represented in tabular form. But we calculate Probability in some given interval say b/w a & b.
i.e,

$$f(x) = \int_a^b f(x)dx$$

A Probability density function has the following Properties

i) $f(x) \geq 0$

ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

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$\int_a^b f(x)dx = 1$

Example 1) Let x be a r.v with p.d.f

$$f(x) = \begin{cases} cx & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the Following

i) c

ii) $P(\frac{1}{2} < x < 3/2)$

iii) $P(x < 1)$

iv) $P(x > 1)$

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Solution:- Given that function is P.d.F

So,

$$\int_0^2 f(x)\,dx = 1$$

$$\int_0^2 cx\,dx = 1$$

$$c \Big|_0^2 \frac{x^2}{2} = 1$$

$$2c = 1$$

$$c = 1/2$$

Now Required P.d.F will be

$$f(x) = \begin{cases} x/2 & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

ii) $P[1/2 < x < 3/2] = \int_{1/2}^{3/2} f(x)\,dx$

$$= \int_{1/2}^{3/2} x/2\,dx$$

$$= \Big|_{1/2}^{3/2} \frac{x^2}{4}$$

$$= \frac{1}{4}\left[\frac{9}{4} - \frac{1}{4}\right]$$

$$= \frac{1}{8}$$

$$= 0.5$$

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iii) $P[x<1] = \int_0^1 f(x)\,dx$

$= \int_0^1 x/2 \, dx$

$= \Big|_0^1 x^2/4$

$= \frac{1}{4} - 0$

$= 0.25$

iv) $P[x>1] = \int_1^2 f(x)\,dx$

$= \int_1^2 x/2 \, dx$

$= \Big|_1^2 x^2/4$

$= 1 - \frac{1}{4}$

$= 0.75$ Required.

Example 2) A continous random variable x has p.d.f

$f(x) = A(2-x)(2+x) \quad 0 \le x \le 2$

find i) A ii) $P(x \le 1)$

iii) $P(1 \le x \le 2)$

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Solution:- Given that function is p.d.f

so,

$$\int_0^2 f(x)dx = 1$$

$$\int_0^2 A(4-x^2) = 1$$

$$A \left| 4x - \frac{x^3}{3} \right|_0^2 = 1$$

$$A\left(8 - \frac{8}{3}\right) = 1$$

$$A\left(\frac{24-8}{3}\right) = 1$$

$$A\left(\frac{16}{3}\right) = 1$$

$$A = 3/16$$

So, Required p.d.f will be

$$f(x) = 3/16(4-x^2)$$

ii) $P(x \leq 1) = \int_0^1 3/16\,(4-x^2)dx$

$$= \frac{3}{16}\left| 4x - \frac{x^3}{3} \right|_0^1$$

$$= \frac{3}{16}\left(4 - \frac{1}{3}\right)$$

$$= \frac{11}{16}$$

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iii) $P(1 \le x \le 2) = \int_1^2 f(x)\,dx$

$= \frac{3}{16}\int_1^2 (4 - x^2)$

$= \frac{3}{16}\Big|_1^2 \left(4x - \frac{x^3}{3}\right)$

$= \frac{3}{16}\left[\left(4(2) - \frac{8}{3}\right) - \left(4 - \frac{1}{3}\right)\right]$

$= \frac{3}{16}\left[8 - \frac{8}{3} - \frac{11}{3}\right]$

$= \frac{3}{16}\left[\frac{24 - 8 - 11}{3}\right]$

$= \frac{5}{16}$ Required.

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Example 3) A Continous r.v can assumes the values b/w $x = 2$ & $x = 4$ has a p.d.f $f(x) = 1/3$ then

i) Show that Area under the curve is 1

ii) $P(1.5 < x < 3)$

Solution:- i)

$\int^4 f(x)\,dx = \frac{1}{3}\int_1^4 dx$

$= \frac{1}{3}\Big|_1^4 x$

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$= \frac{1}{3}(4-1)$

$= \frac{3}{3}$

$= 1$

Hence Proved Area under the curve is 1.

ii) $P(1.5 < x < 3) = \int_{1.5}^{3} f(x)\,dx$

$= \frac{1}{3} \Big| x \Big|_{1.5}^{3}$

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$= \frac{1}{3}(3-1.5)$

$= \frac{1.5}{3}$ Required.

Example 4) Show that function

$f(x) = \frac{1}{18}(3+2x)$ is p.d.f ; $2 \leq x \leq 4$

Solution:- $\int_{2}^{4} f(x)\,dx = \int_{2}^{4} \frac{1}{18}(3+2x)\,dx$

$= \frac{1}{18} \Big| 3x + x^2 \Big|_{2}^{4}$

$= \frac{1}{18}(12+16-6-4)$

$= \frac{18}{18} = 1$ Required.

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"MATHEMATICAL Expectation"

Suppose a r.V x assumes the values $x_1, x_2, x_3, \ldots\ldots, x_n$ with Probabilities $f(x_1), f(x_2), \ldots\ldots, f(x_n)$ then Mathematical Expectation or Expected Value of X denoted by E(X) & is defined by,

$$E(X) = \sum_{i=1}^{n} x_i f(x_i) \quad (\text{For Discrete r.V})$$

$$E(X) = \int_{-\infty}^{\infty} x f(x) \quad (\text{For Continous r.V})$$

Laws of Expectations

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1) $E(C) = C$

2) $E(aX) = a E(X)$

3) $E(X+a) = E(X) + a$

4) $E(X+Y) = E(X) + E(Y)$

5) $E(X-Y) = E(X) - E(Y)$

6) $E(XY) = E(X).E(Y)$

where X & Y are independent.

Problem: Let x be a r.v with p.d.f

x	$f(x)$
-3	1/10
-1	4/10
1	3/10
3	2/10

Find i) $E(X)$ ii) $E(3X)$

iii) $E(X^2)$ iv) $E(3x+5)$

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Solution:-

x	$f(x)$	$3x$	$3xf(x)$	$x^2f(x)$	$3x+5$	$(3x+5)f(x)$
-3	1/10	-9	-9/10	9/10	-4	-4/10
-1	4/10	-3	-12/10	4/10	2	8/10
1	3/10	3	9/10	3/10	8	24/10
3	2/10	9	18/10	18/10	14	28/10

$\Sigma f(x) = 1$ $\Sigma 3xf(x) = 6/10$ $\Sigma x^2f(x) = 34/10$ $\Sigma(3x+5)f(x) = 56/10$

i) $E(X) = \Sigma x f(x) = 2/10 = 0.2$

ii) $E(3X) = \Sigma 3x f(x) = 6/10 = 0.6$

iii) $E(X^2) = \Sigma x^2 f(x) = 34/10 = 3.4$

iv) $E(3x+5) = \Sigma (3x+5) f(x) = 56/10 = 5.6$

Required.

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Remarks:- i) Mean = $E(X)$

ii) $Var(X) = E(X^2) - [E(X)]^2$

Problem:- A r.v "x" has the following p.d.f

X	110	118	120	204
f(x)	0.1	0.3	0.5	0.1

then find

Mean = ?

Var = ?

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Solution:-

X	f(x)	x f(x)	x^2	$x^2 f(x)$
110	0.1	11	12100	1210
118	0.3	35.4	13924	4177.2
120	0.5	60	14400	7200
204	0.1	20.4	41616	4161.6

$\Sigma x f(x) = 126.8$ $\quad \Sigma x^2 f(x) = 16748.8$

Mean = $E(X) = \Sigma x \cdot f(x) = 126.8$

Var $= E(X^2) - [E(X)]^2$

$= \Sigma x^2 f(x) - [\Sigma x f(x)]^2$

Var = 16748.8 − $(126.8)^2$

Var = 670.56 Required.

Problem:- In a summer seasons a dealer of desert roomcoolers can earn RS. 800 per day if the day is hot & can earn RS 200 perday if it is fair & loses RS 50 perday if it is cloudly. Find his Expection if probability of the day being hot is 0.40 & for cloudly it is 0.35

Solution:-

X	P(X)	X P(X)
800	0.40	320
200	0.25	50
−50	0.35	−17.5

$\Sigma X P(X) =$

1 − 0.35 + 0.40 = 0.25

$E(X) = \Sigma X P(X)$

$E(X) = 352.5$ Required.

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Problem: A man draws 2 balls from a bag containing 3 white & 5 black balls. If he is to receive RS 70 for every white balls which he draws & RS 7 for every black ball. Find his Expectation.

White	Black	Total
3	5	8

Solution:-

$$P[2 \text{ white ball}] = \frac{\binom{3}{2}\binom{5}{0}}{\binom{8}{2}} = 3/28$$

$$P[2 \text{ black ball}] = \frac{\binom{3}{0}\binom{5}{2}}{\binom{8}{2}} = 10/28$$

$$P[1 \text{ white } \& \text{ 1 black}] = \frac{\binom{3}{1}\binom{5}{1}}{\binom{8}{2}} = 15/28$$

X	P(X)	X P(X)
140	3/28	15
14	10/28	5
77	15/28	41.25

$$\Sigma X P(X) = 61.25$$

$$E(X) = \Sigma X P(X) = 61.25 \text{ Required.}$$

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Moments Generating Function

The moment Generating Function is usually denoted by $M_x(t)$ of a r.v x about the origin is defined as " The Expected value of the r.v e^{tx} where t is a real number.

$$M_x(t) = M_o(t) = E(e^{tx})$$

$$= \sum e^{tx}(f(x)) \quad \text{For Discrete Random Variable}$$

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$$\int_{-\infty}^{\infty} e^{tx} f(x)\,dx \quad \text{For Continous random Variable.}$$

Moment Generating Function about Origin

$$M_x(t) = M_o(t) = E(e^{tx})$$

$$= \sum_{i=1}^{\infty} e^{tx_i} f(x_i)$$

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= \sum_{i=1}^{\infty} \left(1 + tx_i + \frac{t^2 x_i^2}{2!} + \frac{t^3 x_i^3}{3!} + \cdots\right) f(x_i)$$

$$= \sum_{i=1}^{\infty} f(x_i) + t \sum_{i=1}^{\infty} x_i f(x_i) + \frac{t^2}{2!} \sum_{i=1}^{\infty} x_i^2 f(x_i) + \cdots$$

$$= 1 + t\,E(x_i) + \frac{t^2}{2!} E(x_i^2) + \cdots$$

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$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$$

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Remark:- $\mu_0' = 1$

We find that the co-efficient of $\frac{t^r}{r!}$ in the expression of $E(e^{tx})$ is just $E(x^r)$.

OR

μ_r' the rth moment about origin.

Moment Generating Function about Mean:-

$$M_{\bar{x}}(t) = E\left(e^{t(x-\bar{x})}\right)$$

$$= E(e^{tx} \cdot e^{-t\bar{x}})$$

$$= e^{-t\bar{x}} E(e^{tx}) \qquad \because E(ax) = aE(x)$$

$$= e^{-t\bar{x}} M_x(t)$$

OR

$$= e^{-t\bar{x}} M_0(t)$$

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Moment Generating Function about "a"

$$M_a(t) = E\left(e^{t(x-a)}\right)$$

$$= E\left(e^{tx} \cdot e^{-ta}\right)$$

$$= e^{-ta} E\left(e^{tx}\right)$$

$$= e^{-ta} M_x(t) \quad \text{OR}$$

$$= e^{-ta} M_0(t)$$

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Problem:- (1999, 2000) Prove that

$$M_{\frac{x-a}{b}}(t) = e^{-at/b} M_x(t/b)$$

$$M_{\frac{x-a}{b}}(t) = E\left[e^{t\left(\frac{x-a}{b}\right)}\right]$$

$$= E\left[e^{tx/b} \cdot e^{-at/b}\right]$$

$$= e^{-at/b} E\left(e^{tx/b}\right) \quad \because E(ax) = aE(x)$$

$$= e^{-at/b} M_x(t/b)$$

which is Required. Proved.

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Theorem:- Show that Moment Generating function of sum of two independent random variables is product of their respective Moment Generating Functions.

Solution:- let X & Y be two indep r.v.

$$M_X(t) = E(e^{tX}) \rightarrow ①$$

$$M_Y(t) = E(e^{tY}) \rightarrow ②$$

Now,

$$M_{X+Y}(t) = E\left[e^{t(X+Y)}\right]$$

$$= E\left[e^{tX} \cdot e^{tY}\right]$$

$$= E(e^{tX}) \cdot E(e^{tY}) \quad \because X \,\&\, Y \text{ are indp}$$

by using ① & ②

$$M_{X+Y}(t) = M_X(t) \, M_Y(t)$$

As Required. Proof

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Problem:- Find K, Mean & Mode for the distribution whose p.d.f is

$$f(x) = k(6 + x - x^2) \qquad 0 \leq x \leq 3$$

Also determine its Var.

Solution:- i) $K = ?$

As function is p.d.f so,

$$\int_0^3 f(x)dx = 1$$

$$\int_0^3 k(6 + x - x^2) = 1$$

$$k \left| 6x + \frac{x^2}{2} - \frac{x^3}{3} \right|_0^3 = 1$$

$$k \left[18 + \frac{9}{2} - 9 \right] = 1$$

$$k \left[9 + \frac{9}{2} \right] = 1$$

$$k \left[\frac{18 + 9}{2} \right] = 1$$

$$k \left[\frac{27}{2} \right] = 1$$

$$\Rightarrow k = 2/27$$

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ii)

Now $f(x) = \frac{2}{27}(6+x-x^2)$

$E(X) = \text{Mean} = \int_a^b x f(x)\,dx$

$= \int_0^3 x\left(\frac{2}{27}(6+x-x^2)\right)dx$

$= \frac{2}{27}\int_0^3 (6x+x^2-x^3)\,dx$

$= \frac{2}{27}\left|\frac{6x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}\right|_0^3$

$= \frac{2}{27}\left[27+9-\frac{81}{4}\right]$

$= \frac{2}{27}\left[36-\frac{81}{4}\right]$

$= \frac{2}{27}\left[\frac{144-81}{4}\right]$

$= \frac{2}{27}\left[\frac{63}{4}\right]$

$= \frac{63}{54}$

$= 1.16$

Required.

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Now

$$E(X^2) = \int_0^3 x^2 f(x)\,dx$$

$$= \int_0^3 x^2 \cdot 2/27\,(6 + x - x^2)\,dx$$

$$= 2/27 \int_0^3 (6x^2 + x^3 - x^4)\,dx$$

$$= 2/27 \Big|_0^3 \frac{6x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5}$$

$$= 2/27 \left[54 + \frac{81}{4} - \frac{243}{5} \right]$$

$$= 2/27 \left[\frac{1080 + 405 - 972}{20} \right]$$

$$= \frac{2}{27} \cdot \frac{513}{20}$$

$$= \frac{513}{270}$$

$$= 1.9$$

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$$Var = E(X^2) - [E(X)]^2$$

$$= 1.9 - (1.16)^2$$

$$= 1.9 - 1.34$$

$$= -0.55$$

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For Mode

$$f(x) = \frac{2}{27}(6+x-x^2)$$

$$f'(x) = \frac{2}{27}(1-2x)$$

Put $f'(x) = 0$

$$1-2x = 0$$

$$x = \frac{1}{2}$$

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Now

$$f''(x) = -\frac{4}{27}$$

at $x = \frac{1}{2}$, $f''(x) < 0$

$\Rightarrow$ Mode $= \frac{1}{2}$ Required.

Problem:- A dist is given by

$$f(x) = x^2(1-x) \quad \text{b/w } x = 0 \text{ \& } x = 1$$

Find Mean & Standard Deviation.

Solution:- First of all we check wheater the given function is p.d.f is not,

$$\int_0^1 f(x)dx = \int_0^1 (x^2 - x^3)\, dx$$

$$= \Big|_0^1 \frac{x^3}{3} - \frac{x^4}{4}$$

$$= \frac{1}{3} - \frac{1}{4}$$

$$= \frac{1}{12}$$

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Function will be p.d.f when we multiply the function with 12

i.e, Required p.d.F is,

$$f(x) = 12x^2(1-x) \quad ; \quad 0 \le x \le 1$$

$$\text{Mean} = E(X) = \int_0^1 x\, f(x)dx$$

$$= 12\int_0^1 (x^3 - x^4)\, dx$$

$$= 12 \Big|_0^1 x^4/4 - x^5/5$$

$$= 12\left[\frac{1}{4} - \frac{1}{5}\right] \Rightarrow \frac{12}{20} = 3/5$$

$\therefore$ Mean $= 3/5$ Required.

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Now,

$$E(X^2) = \int_0^1 x^2 f(x)dx$$

$$= 12\int_0^1 x^2 \; x^2(1-x)\,dx$$

$$= 12 \int_0^1 (x^4 - x^5)\,dx$$

$$= 12 \left| \frac{x^5}{5} - \frac{x^6}{6} \right|_0^1$$

$$= 12\left[\frac{1}{5} - \frac{1}{6}\right] = \frac{12}{30} = 2/5$$

Now,

$$S.Dev = \sqrt{E(X^2) - (E(X))^2}$$

$$= \sqrt{2/5 - (3/5)^2}$$

$$= \sqrt{2/5 - 9/25}$$

$$= \sqrt{\frac{10 - 9}{25}}$$

$$= \sqrt{1/25}$$

$$= 1/5 \text{ Required.}$$

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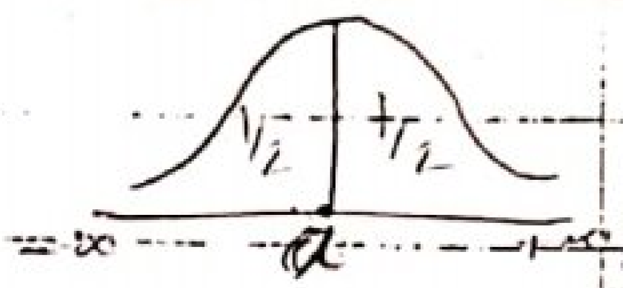
Question:- A Random Variable x has the density function given by,

$$f(x) = \begin{cases} 2(1-x) & 0 \leq x \leq 1 \\ 0 & \text{Elsewhere} \end{cases}$$

Find Mean, Mode, Lower Quartile & Upper Quartile.

Solution:- Median is defined as,



$$\int_{-\infty}^{a} f(x)dx = 1/2$$

where value of a is Median.

$$\int_{0}^{a} 2(1-x)dx = 1/2$$

$$2 \left| x - x^2/2 \right|_0^a = 1/2$$

$$2(a - a^2/2) = 1/2$$

$$2a - a^2 = 1/2$$

$$4a - 2a^2 = 1$$

$$2a^2 - 4a + 1 = 0$$

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$$a = \frac{+4 \pm \sqrt{16 - 8}}{2 \times 2}$$

$$a = \frac{4 \pm \sqrt{8}}{4}$$

$$a = \frac{4 \pm 2\sqrt{2}}{4}$$

$$a = 1 \pm \frac{\sqrt{2}}{2}$$

$$a = 1 \pm \sqrt{1/2}$$

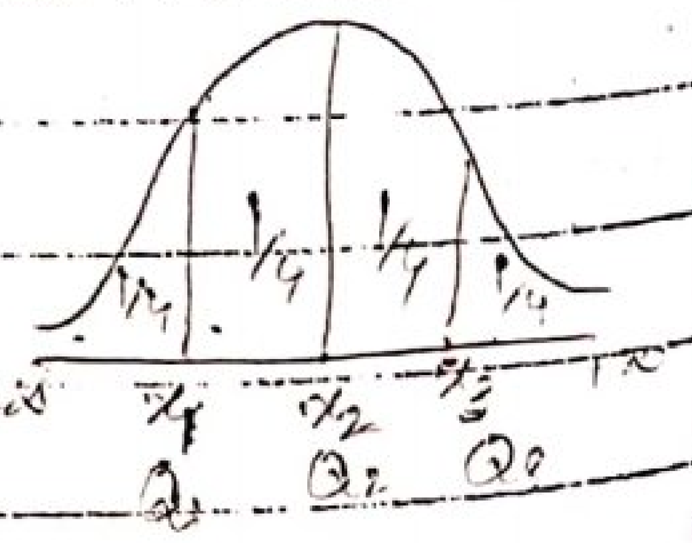
$\Rightarrow$ Median $= 1 - \sqrt{1/2}$ $\because$ $x \in [0, 1]$

Now we calculate Mode,

$$f(x) = 2 - 2x$$

$$f'(x) = -2$$

So, Mode does not Exist.



Now,

$$Q_1 = \int_{-\infty}^{a} f(x)\,dx = 1/4$$

where the value of a is Q_1.

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$$2\int_0^a (1-x)\,dx = 1/4$$

$$2\Big|_0^a x - \frac{x^2}{2} = 1/4$$

$$2\left[a - \frac{a^2}{2}\right] = 1/4$$

$$2a - \frac{2a^2}{2} = 1/4$$

$$8a - 4a^2 = 1$$

$$4a^2 - 8a + 1 = 0$$

$$a = \frac{8 \pm \sqrt{64-16}}{2 \times 4}$$

$$a = \frac{8 \pm \sqrt{48}}{8}$$

$$a = \frac{8 \pm 4\sqrt{3}}{8}$$

$$a = 1 \pm \sqrt{3}/2$$

$$a = 1 - \sqrt{3}/2 \quad \because x \in [0,1]$$

$\therefore \quad Q_1 = 1 - \sqrt{3}/2$ Required.

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For $Q_3 = \int_{-\infty}^{a} f(x)dx = 3/4$

where a is Q_3.

$$2\int_0^a (1-x)dx = 3/4$$

$$2 \left| x - \frac{x^2}{2} \right|_0^a = 3/4$$

$$2\left[a - \frac{a^2}{2}\right] = 3/4$$

$$2a - a^2 = 3/4$$

$$8a - 4a^2 = 3$$

$$4a^2 - 8a + 3 = 0$$

$$a = \frac{8 \pm \sqrt{64-48}}{2 \times 4}$$

$$= \frac{8 \pm \sqrt{16}}{8}$$

$$= \frac{8 \pm 4}{8} \Rightarrow \frac{12}{8}, \frac{4}{8}$$

$\therefore a = 1/2 \quad \because x \in [0,1]$

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Problem:- A random variable "x" has p.d.f

$f(x) = 2(1-x) \quad ; \quad 0 < x < 1$

Find $E(X)$, $E(X^2)$, $Var(X)$, $E(X+10)^2$.

Solution:- i) $E(X) = \int_a^b x\, f(x)\, dx$

$E(X) = \int_0^1 x\, 2(1-x)\, dx$

$= 2 \int_0^1 x - x^2$

$= 2 \left| \frac{x^2}{2} - \frac{x^3}{3} \right|_0^1$

$E(X) = 2\left(\frac{1}{2} - \frac{1}{3}\right) = 2\left(\frac{1}{6}\right)$

$E(X) = 1/3$

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ii) $E(X^2) = \int_a^b x^2 f(x)\, dx$

$= 2\int_0^1 x^2(1-x)\, dx$

$= 2 \int_0^1 x^2 - x^3$

$= 2 \left| \frac{x^3}{3} - \frac{x^4}{4} \right|_0^1 \Rightarrow 2\left\{\frac{1}{3} - \frac{1}{4}\right\}$

$= \frac{1}{6}$.

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iii) $Var(X) = E(X^2) - [E(X)]^2$

$= \frac{1}{6} - \left(\frac{1}{3}\right)^2$

$= \frac{1}{6} - \frac{1}{9}$

$= \frac{3-2}{18}$

$= 1/18$ Required.

iv) $E(X+10)^2 = E[X^2 + 100 + 20X]$

$= E(X^2) + E(100) + 20E(X)$

$= \frac{1}{6} + 100 + 20\left(\frac{1}{3}\right)$

$= \frac{1}{6} + 100 + \frac{20}{3}$

$= \frac{1 + 600 + 40}{6}$

$= \frac{641}{6}$

Required.

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Problem:- A Continous Random Variable has p.d.f as follows

$$f(x) = \begin{cases} x/2 & 0 < x \leq 1 \\ 1/4(3-x) & 1 < x \leq 2 \\ 1/4 & 2 < x \leq 3 \\ 1/4(4-x) & 3 < x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

Find, i) $P(x > 3)$ ii) $P(x < 1.5)$

iii) $P(1 < x < 3)$

Solution:- (i) $P(x>3) = \int_3^4 f(x)\,dx$

$$= \int_3^4 1/4(4-x) = \frac{1}{4}\Big|_3^4 4x - x^2/2$$

$$= \frac{1}{4}\left[(16 - \frac{16}{2}) - (12 - 9/2)\right]$$

$$= \frac{1}{4}\left[(16-8) - (\frac{24-9}{2})\right]$$

$$= \frac{1}{4}\left[8 - \frac{15}{2}\right]$$

$$= \frac{1}{4}\left[\frac{16-15}{2}\right]$$

$= 1/8$ Required.

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ii) $P(X<1.5) = \int_0^{1.5} f(x)\,dx$

$= \int_0^1 f(x)\,dx + \int_1^{1.5} f(x)\,dx$

$= \int_0^1 x/2\,dx + \int_1^{1.5} 1/4\,(3-x)\,dx$

$= \left.\frac{x^2}{4}\right|_0^1 + \left.\frac{1}{4}\left(3x - \frac{x^2}{2}\right)\right|_1^{1.5}$

$= \frac{1}{4} + \frac{1}{4}\left\{\left(3(1.5) - \frac{(1.5)^2}{2}\right) - \left(3 - \frac{1}{2}\right)\right\}$

$= \frac{1}{4} + \frac{1}{4}\left[4.5 - \frac{2.25}{2} - 5/2\right]$

$= \frac{1}{4}\left[1 + 4.5 - \frac{2.25}{2} - 5/2\right]$

$= \frac{1}{8}\left[2 + 9 + 2.25 + 5\right]$

$= 1/8\,[3.75]$

$= 3.75/8$ Required.

iii) $P(1<X<3) = \int_1^3 f(x)\,dx$

$= \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx$

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$$= \int_1^2 \tfrac{1}{4}\,dx + \int_2^3 \tfrac{1}{4}(4-x)\,dx$$

$$= \frac{1}{4}\Big|_1^2 x + \frac{1}{4}\Big|_2^3 4x - x^2/2$$

$$= \frac{1}{4}(2-1) + \frac{1}{4}\left[(12 - 9/2) - (8 - 4/2)\right]$$

$$= \frac{1}{4} + \frac{1}{4}\left[12 - 9/2 - 6\right]$$

$$= \frac{1}{4}\left[1 + 6 - 9/2\right]$$

$$= \frac{1}{4}\left[\frac{2 + 12 - 9}{2}\right]$$

$$= \frac{1}{4}\left[\frac{5}{2}\right]$$

$$= 5/8$$ Required.

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Distribution Function:-

The function which gives the Probability that a, r.v X assumes a values less than or equal to the specified value "x" is called Distribution, function is denoted by F(X)

$$F(X) = P(X \leq x)$$

A distribution function have the following Properties,

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i) $F(-\infty) = 0$

ii) $F(+\infty) = 1$

iii) $F(x)$ is non decreasing function.

Problem 1) Three coins are Tossed. Let X denotes the number of heads.
Find
i) Probability Dist of x.
ii) Dist function of x.

Solution:- S = {TTT, TTH, THT, HTT, HHT, THH, HTH, HHH}

(i)

x	P(x)
0	1/8
1	3/8
2	3/8
3	1/8

Required.

ii)

$X \le x$	F(X)
$X \le 0$	1/8
$X \le 1$	4/8
$X \le 2$	7/8
$X \le 3$	8/8

* $\frac{1}{8} + \frac{3}{8} = 4/8$
* $4/8 + 3/8 = 7/8$
* $7/8 + 1/8 = 8/8$

Required.

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Problem 2) In the following Table shows distribution Function F(x) of random variable X.

$X \le x$	$F(X)$
$X \le 1$	1/8
$X \le 2$	3/8
$X \le 3$	3/4
$X \le 4$	1

Find

i) Probability Dist of r.v. X

ii) E(X) & Var(X).

Solution: i)

$X \le x$	$F(X)$	$f(x)$
$X \le 1$	1/8	1/8
$X \le 2$	3/8	2/8
$X \le 3$	3/4 = 6/8	3/8
$X \le 4$	1 = 8/8	2/8

ii)

x	$f(x)$	$x\,f(x)$	$x^2 f(x)$
1	1/8	1/8	1/8
2	2/8	4/8	8/8
3	3/8	9/8	27/8
4	2/8	8/8	32/8

$\Sigma x f(x) = 22/8$ $\quad \Sigma x^2 f(x) = 68/8$

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$E(X) = \sum x f(x) = 22/8$

$Var(X) = E(X^2) - [E(X)]^2$

$= \frac{66}{8} - (22/8)^2$

$= \frac{66}{8} - (\frac{22}{8})(\frac{22}{8})$

$= \frac{22}{8}[3 - 22]$

$= \frac{22}{8}[-19]$

$= \frac{-418}{8} \Rightarrow \frac{-209}{4}$ Required.

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Remark:- In case of continous r.v

$$F(x) = \int_{-\infty}^{x} f(x)\,dx \quad \&$$

$$f(x) = \frac{dF(x)}{dx}$$

Problem:- A continous r.v has a dist function $F(x)$ as follows

$$F(x) = \begin{cases} 0 & x < 0 \\ 2x^2/5 & 0 \leq x \leq 1 \\ -3/5 + 2/5(3x - x^2/5) & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

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Find) i) Probability density function

ii) $P(X < 1.5)$ iii) $E(x)$, iv) $Var(x)$.

Solution:-

$$f(x) = \frac{dF(x)}{dx} = \begin{cases} 0 & x < 0 \\ 4x/5 & 0 \le x < 1 \\ 2/5(3 - 2/5\,x) & 1 < x < 2 \\ 0 & x \geq 2 \end{cases}$$

ii) $P(X < 1.5) = \int_0^{1.5} f(x)\,dx$

$$= \int_0^1 f(x)\,dx + \int_1^{1.5} f(x)\,dx$$

$$= \int_0^{1.5} 4x/5\,dx + \int_1^{1.5} 2/5(3 - 2/5\,x)\,dx$$

$$= \frac{4}{5}\Big|_0^1 x\,dx + 2/5\,\Big|_1^{1.5} 3x - \frac{x^2}{5}$$

$$= \frac{4}{5}(1) + 2/5\left(\left(4.5 - \frac{2.25}{5}\right) - \left(3 - \frac{1}{5}\right)\right)$$

$$= \frac{4}{5} + 2/5\left[\left(\frac{22.5 - 2.25}{5}\right) - \left(\frac{15 - 1}{5}\right)\right]$$

$$= 4/5 + \frac{2}{25}\left[20.25 - 14\right]$$

$$= 4/5 + \frac{12.5}{25} \Rightarrow \frac{32.5}{25}$$ Required.

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(ii) $E(X) = \int_0^2 x\, f(x)\, dx$

$E(X) = \int_0^1 x f(x)\, dx + \int_1^2 x f(x)\, dx$

$= \int_0^1 4x^2/5\, dx + \int_1^2 2/5\,(3x - 2/5\, x^2)\, dx$

$= \frac{4}{5} \left| \frac{x^3}{3} \right._0^1 + \frac{2}{5} \left| \frac{3x^2}{2} - \frac{2x^3}{15} \right._1^2$

$= \frac{4}{5}\left(\frac{1}{3}\right) + \frac{2}{5}\left[\left(6 - \frac{16}{15}\right) - \left(\frac{3}{2} - \frac{2}{15}\right)\right]$

$= \frac{4}{15} + \frac{2}{5}\left[6 - \frac{16}{15} - \frac{3}{2} + \frac{2}{15}\right]$

$= \frac{4}{15} + \frac{2}{5}\left[6 - \frac{3}{2} - \frac{14}{15}\right]$

$= \frac{4}{15} + \frac{2}{5}\left[\frac{180 - 45 - 28}{30}\right]$

$= \frac{4}{15} + \frac{1}{75}\left[107\right]$

$= \frac{20 + 107}{75} \Rightarrow \frac{127}{75}$ Required.

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iv) $E(X^2) = \int_0^2 x^2 f(x)\,dx$

$= \int_0^1 x^2 f(x)\,dx + \int_1^2 x^2 f(x)\,dx$

$= \int_0^1 4x^3/5\,dx + \int_1^2 2/5(3x^2 - 2/5\,x^3)\,dx$

$= \frac{4}{5}\Big|_0^1 \frac{x^4}{4} + 2/5\Big|_1^2 \left(x^3 - \frac{x^4}{10}\right)$

$= \frac{4}{5}\left(\frac{1}{4}\right) + 2/5\left\{\left(8 - \frac{4}{10}\right) - \left(1 - \frac{1}{10}\right)\right\}$

$= \frac{1}{5} + 2/5\left\{8 - 2/5 - 1 + 1/10\right\}$

$= 1/5 + 2/5\left\{7 - 2/5 + 1/10\right\}$

$= 1/5 + 2/5\left\{\frac{70 - 4 + 1}{10}\right]$

$= 1/5 + \frac{1}{25}\left\{65\right\}$

$= \frac{1}{5} + \frac{13}{5}$

$= \frac{14}{5}$

Now $Var(X) = E(X^2) - [E(X)]^2$

$= \frac{14}{5} - \left(\frac{127}{75}\right)^2$

$= \frac{15750 - 16129}{(75)^2} = \frac{-379}{(75)^2}$

Required.

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Problem:- If p.d.f of x is given by

$$f(x) = \begin{cases} c/\sqrt{x} & 0 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

Find)
i) $c = ?$
ii) distribution function
iii) $P(X > 1)$

Solution:-

$$\int_0^4 c/\sqrt{x}\, dx = 1$$

$$c \Big|_0^4 2\sqrt{x} = 1$$

$$c(4) = 1$$

$$c = 1/4$$

$\therefore$

$$f(x) = \frac{1}{4\sqrt{x}} \quad 0 < x < 4$$

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ii) $F(x) = \int_0^x f(x)dx$

$= \int_0^x \frac{1}{4\sqrt{x}} dx$

$= \sqrt{x}/2$

iii) $P(X>1) = \int_1^4 f(x)dx$

$= \int_1^4 \frac{1}{4\sqrt{x}} dx$

$= \frac{1}{4} \int_1^4 (x)^{-1/2} dx$

$= \frac{1}{4} \Big|_1^4 \frac{x^{1/2}}{1/2}$

$= \frac{1}{2} \{ 2 - 1]$

$= 1/2$ Required.

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Problem:- For what value of K the following function is a density function

$f(x) = Kx^2(1-x) \quad 0 \le x < 1$

Find i) Distribution Function
ii) Mean iii) Var(x)

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Solution:- $\int_0^1 f(x)\,dx = 1$

$\int_0^1 k\, x^2(1-x)\,dx = 1$

$k \left| -\frac{x^4}{4} + \frac{x^3}{3} \right|_0^1 = 1$

$k \left\{ \frac{1}{3} - \frac{1}{4} \right\} = 1$

$k \left\{ \frac{1}{12} \right\} = 1$

$k = 12$

So,

$f(x) = 12x^2(1-x) \quad ; \quad 0 \leq x \leq 1$

(i) $F(x) = \int_0^x f(x)\,dx$

$= 12 \int_0^x x^2(1-x)$

$= 12 \left| \frac{x^3}{3} - \frac{x^4}{4} \right|_0^x$

$= 12 \left\{ \frac{4x^3 - 3x^4}{12} \right\}$

$= 4x^3 - 3x^4$

$F(x) = x^3(4-3x)$ Required.

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iii) Mean = $E(X) = \int_0^1 x f(x) dx$

$= \int_0^1 x \cdot 12 x^2 (1-x) dx$

$= 12 \int_0^1 (x^3 - x^4) dx$

$= 12 \left| \frac{x^4}{4} - \frac{x^5}{5} \right|_0^1$

$= 12 \left(\frac{1}{4} - \frac{1}{5} \right)$

$= 12 (1/4 - 1/5)$

$= 12 \left(\frac{5-4}{20} \right)$

$= \frac{12}{20} \Rightarrow 6/10 = 3/5$

$E(X^2) = \int_0^1 x^2 f(x) dx$

$= \int_0^1 12 x^4 (1-x) dx$

$= 12 \left| \frac{x^5}{5} - \frac{x^6}{6} \right|_0^1$

$= 12 \left(\frac{1}{5} - \frac{1}{6} \right) \Rightarrow \frac{12}{30} \Rightarrow 2/5$

$Var(X) = E(X^2) - [E(X)]^2$

$= 2/5 - 9/25 = \frac{10-9}{25} \Rightarrow 1/25$

Required.

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Assignment # 4

Question 1) a) Given a Random Variable X with $E(X) = 0.63$ & $Var(X) = 0.2331$

$E(X^2) = ?$

Solution: As we know that

$$Var(X) = E(X^2) - [E(X)]^2$$

$$\therefore E(X^2) = Var(X) + [E(X)]^2$$

$$= 0.2331 + 0.3969$$

$$E(X^2) = 0.63 \quad \text{Required.}$$

b) Given that $E(X^2) = 400$ & $S.D(X) = 12$

Find $\Rightarrow E(X)$.

As we know,

$$S.D(X) = \sqrt{E(X^2) - [E(X)]^2}$$

$$12 = \sqrt{400 - [E(X)]^2}$$

$$144 = 400 - [E(X)]^2$$

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$[E(X)]^2 = 400 - 144$

$[E(X)]^2 = 256$

$E(X) = \sqrt{256}$

$E(X) = 16$ Required.

c) $E(X) = 200$, $C.V(X) = 7\%$, $Var(X) = ?$

$C.V = \frac{S}{\bar{x}} \times 100$ ✓

$7 = \frac{S}{200} \times 100$

$7 = \frac{S}{2}$

$S = 14$

$var(x) = (14)^2 =$

$Var(x) = 196$

As Required.

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Question 2) If $f(x) = \frac{6 - |7-x|}{36}$ for $x = 2, 3, \ldots, 12$

find Mean & Variance of X.

Solution:-

x	$f(x)$	$xf(x)$	x^2	$x^2 f(x)$
2	1/36	2/36	4	4/36
3	2/36	6/36	9	18/36
4	3/36	12/36	16	46/36
5	4/36	20/36	25	100/36
6	5/36	30/36	36	180/36
7	6/36	42/36	49	294/36
8	5/36	40/36	64	320/36
9	4/36	36/36	81	324/36
10	3/36	30/36	100	300/36
11	2/36	22/36	121	242/36
12	1/36	12/36	144	144/36

$\Sigma x f(x) = 252/36$ $\quad\quad$ $\Sigma x^2 f x = \frac{1974}{36}$

$$\text{Mean} = \Sigma x f(x) = 252/36 = 42/6 = 7$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{1974}{36} - 49$$

$$= \frac{1974 - 1764}{36}$$

$$= \frac{210}{36} = \frac{70}{12} = 35/6 \text{ - Required.}$$

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Question 3) A Random Variable X has the probability distribution

$$f(x) = K\binom{5}{x} \quad x = 0, 1 \text{ --- }, 5$$

Find i) K ii) $E(X)$ iii) $Var(X)$.

Solution:

x	$f(x)$	$xf(x)$	x^2	$x^2 f(x)$
0	1/32	0	0	0
1	5/32	5/32	1	5/32
2	10/32	20/32	4	40/32
3	10/32	30/32	9	90/32
4	5/32	20/32	16	80/32
5	1/32	5/32	25	25/32

i) $K\binom{5}{0} + K\binom{5}{1} + K\binom{5}{2} + K\binom{5}{3} + K\binom{5}{4} + K\binom{5}{5} = 1$

$$K(1) + K(5) + K(10) + K(10) + K(5) + K = 1$$

$$K[1+5+10+10+5+1] = 1$$

$$K[32] = 1$$

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$$K = 1/32$$

So, $f(x) = \frac{1}{32}\binom{5}{x}$; $x = 0, 1, 2, \text{ --- } 5$

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$E(X) = \Sigma x f(x)$

$= 80/32 = 20/8 = 5/2$

$E(X^2) = \Sigma x^2 f(x)$

$= \frac{240}{32} = \frac{30}{4} \Rightarrow 15/2$

Now,

$Var(X) = E(X^2) - [E(X)]^2$

$= 15/2 - 25/4$

$= \frac{30 - 25}{4}$

$= 5/4$ Required.

Question 4) a) A Committe of Size 5 is to be selected at random from 3 women & 5 men. Find Expected number of women on the committee.

Solution:-

Women	Men	Total
3	5	8

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x	$f(x)$	$f(x)$	$x f(x)$
0	$\binom{3}{0}\binom{5}{5}/\binom{8}{5}$	$1/56$	0
1	$\binom{3}{1}\binom{5}{4}/\binom{8}{5}$	$15/56$	$15/56$
2	$\binom{3}{2}\binom{5}{3}/\binom{8}{5}$	$30/56$	$60/56$
3	$\binom{3}{3}\binom{5}{2}/\binom{8}{5}$	$10/56$	$30/56$
		$\Sigma f(x) = 1$	$\Sigma x f(x) = 105/56$

$E(X) = \Sigma x f(x)$

$E(X) = 105/56$

$E(X) = 15/8$

$E(X) = 1.875$

AS. Required.

* Work Hard & All left on All mighty ALLAH. *

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b) If it rains a raincoat dealer can earn Rs 500 perday. If it is fair he can lose Rs 100 per day what is his expectation if the probability of Rain is 0.4.

Solution:-

X	f(x)	x f(x)
500	0.4	200
-100	0.6	-60
		$\Sigma x f(x) = 140$

$E(X) = \Sigma x f(x)$

$E(X) = 140$

Required.

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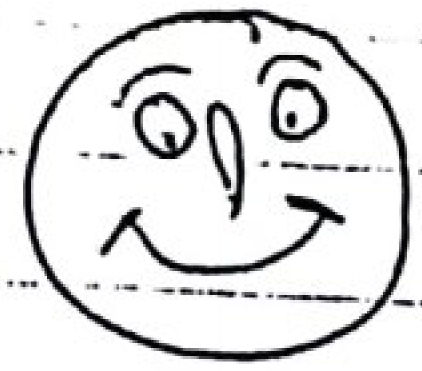
Question 5) :- Check in Notes.



HA,HA.....

Wish You Good Luck!

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CHAPTER # 5

Discrete Probability Distribution

1) Discrete Uniform Distribution:-

The Discrete Uniform Distribution arises when the underlying random Variable can take "k" different Values with equal Probabilities.

In the special case when random variate X can take values

$X = 1, 2, \ldots\ldots, K$ The Possibility Dist is given by,

$$f(x) = f(x;k) = \begin{cases} 1/k & , \quad x = 1, 2, 3, \ldots, k \\ 0 & \quad \text{otherwise} \end{cases}$$

The Discrete uniform Dist some time given by,

$$f(x) = f(x;k) = 1/_{k+1} \; ; x = 0, 1, 2, \ldots, k-1$$

Question:- Find Mean & Variance of Discrete Uniform Distribution.

Solution:- In case of Discrete uniform Distribution,

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$f(x) = 1/k \quad ; x = 1, 2, \ldots, k$

$$\text{Mean} = E(X) = \sum_{x=1}^{k} x f(x)$$

$$= \frac{1}{k} \sum_{x=1}^{k} x$$

$$= \frac{1}{k}\left[1 + 2 + 3 + \cdots + k\right]$$

$$= \frac{1}{k} \cdot \frac{k(k+1)}{2}$$

$$E(X) = \frac{k+1}{2}$$

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Now

$$E(X^2) = \sum_{x=1}^{k} x^2 f(x)$$

$$= \frac{1}{k} \sum_{x=1}^{k} x^2$$

$$= \frac{1}{k}\left[1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2\right]$$

$$= \frac{1}{k} \cdot \frac{k(k+1)(2k+1)}{6}$$

$$= \frac{(k+1)(2k+1)}{6}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \frac{(k+1)(2k+1)}{6} - \frac{(k+1)^2}{4}$$

$$= \frac{(K+1)}{2}\left[\frac{2K+1}{3} - \frac{K+1}{2}\right]$$

$$= \frac{K+1}{2}\left[\frac{4K+2-3K-3}{6}\right]$$

$$= \frac{K+1}{2}\left[\frac{K-1}{6}\right]$$

$$= \frac{K^2-1}{12}$$ Required.

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Problem 2) Derive Moment generating function of Discrete uniform dist use it to find four moments of Discrete uniform distribution, also using these Moments find moments about Mean.

Solution:- As we know

$$M_X(t) = E(e^{tX})$$

$$= \sum_{x=1}^{K} e^{tx} f(x)$$

$$= \frac{1}{K}\sum_{x=1}^{K} e^{tx}$$

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$$M_X(t) = \frac{1}{K}\left[e^t + e^{2t} + e^{3t} + \cdots + e^{Kt}\right]$$

which m.g.f of discrete uniform distribution.

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$$\mu_1' = \frac{d}{dt}\left[M_X(t)\right]_{t=0} = \frac{1}{k}\left[e^t + 2e^{2t} + 3e^{3t} + \cdots + ke^{kt}\right]_{t=0}$$

$$\mu_1' = \frac{1}{k}\left[1 + 2 + 3 + \cdots + k\right]$$

$$\mu_1' = \frac{1}{k} \cdot \frac{k(k+1)}{2}$$

$$\mu_1' = \frac{k+1}{2}$$

$$\mu_2' = \frac{d^2}{dt^2}\left[M_X(t)\right]_{t=0} = \frac{1}{k}\left[e^t + 4e^{2t} + 9e^{3t} + \cdots + k^2 e^{kt}\right]$$

$$\mu_2' = \frac{1}{k}\left[1^2 + 2^2 + 3^2 + \cdots + k^2\right]$$

$$\mu_2' = \frac{1}{k} \cdot \frac{k(k+1)(2k+1)}{6}$$

$$\mu_2' = \frac{(k+1)(2k+1)}{6}$$

$$\mu_3' = \frac{d^3}{dt^3}\left[M_X(t)\right]_{t=0} = \frac{1}{k}\left[e^t + 8e^{2t} + 27e^{3t} + \cdots + k^3 e^{kt}\right]$$

$$\mu_3' = \frac{1}{k}\left[1^3 + 2^3 + 3^3 + \cdots + k^3\right]$$

$$\mu_3' = \frac{1}{k}\left[\frac{k(k+1)}{2}\right]^2$$

$$\mu_3' = \frac{k(k+1)^2}{4}$$

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$$\mu_4' = \frac{d^4}{dt^4}\left[M_X(t)\right]_{t=0} = \frac{1}{k}\left[e^t + 16e^{2t} + 81e^{3t} + \cdots + k^4 e^{kt}\right]$$

$$\mu_4' = \frac{1}{k}\left[1^4 + 2^4 + 3^4 + \cdots + k^4\right]$$

$$\mu_4' = \frac{1}{k} \cdot \frac{k(6k^4 + 15k^3 + 10k^2 - 1)}{30}$$

$$\mu_4' = \frac{6k^4 + 15k^3 + 10k^2 - 1}{30}$$

As Required.

Moments About Mean:-

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$= \frac{(k+1)(2k+1)}{6} - \frac{(k+1)^2}{4}$$

$$= \frac{k+1}{2}\left[\frac{2k+1}{3} - \frac{k+1}{2}\right]$$

$$= \frac{k+1}{2}\left[\frac{4k+2-3k-3}{6}\right]$$

$$= \frac{k+1}{2}\left[\frac{k-1}{6}\right]$$

$$\mu_2 = \frac{(k+1)(k-1)}{12} \Rightarrow \frac{k^2-1}{12}$$

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$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2\mu_1'^3$$

$$\mu_3 = \frac{k(k+1)^2}{4} - 3\left(\frac{k+1}{2}\right)\left(\frac{(k+1)(2k+1)}{6}\right) + 2\left(\frac{k+1}{2}\right)^3$$

$$\mu_3 = \frac{(k+1)^2}{4}\left[k - (2k+1) + (k+1)\right]$$

$$\mu_3 = \frac{(k+1)^2}{4}\left[k - 2k - 1 + k + 1\right]$$

$$\mu_3 = 0$$

$$\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6\mu_1'^2\mu_2' - 3\mu_1'^4$$

$$= \frac{6k^4 + 15k^3 + 10k^2 - 1}{30} + 6\left(\frac{k+1}{2}\right)^2\left(\frac{(k+1)(2k+1)}{6}\right) - 4\left(\frac{k+1}{2}\right)\left(\frac{k(k+1)^2}{4}\right) - 3\left(\frac{k+1}{2}\right)^4$$

$$= \frac{6k^4 + 15k^3 + 10k^2 - 1}{30} + \frac{(k+1)^3}{4}\left[2k + 1 - 2k - 3\,\frac{k+1}{4}\right]$$

$$= \frac{6k^4 + 15k^3 + 10k^2 - 1}{30} + \frac{(k+1)^3}{16}\left[4 - 3k - 3\right]$$

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$$= \frac{6k^4+15k^3+10k^2-1}{30} + \frac{(k+1)^3(1-3k)}{16}$$

$$= \frac{48k^4+120k^3+80k^2-8+15-45k^4-120k^3-90k^2}{240}$$

$$= \frac{3k^4-10k^2+7}{240}$$

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$$= \frac{(k^2-1)(3k^2-7)}{240}$$ Required.

Bernoulli's Trials:- If the Probability remains the same through out the trials, such trials are called Bernoulli's Trials.

Binomial Experiment:- An Exp is called Binomial Exp if it has the following Properties,

1) Exp has only two outcomes, success OR failure.
2) Probability of success remains constant from trail to trial.
3) Trails are independent.
4) Exp is Repeated a fixed no of times say "n".

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Poisson Experiment:- An Exp is called Poisson experiment if it has following properties

i) Exp has only two outcomes, success OR Failure.
ii) Probability of success remains constant from trial to trial.
iii) Trials are independent.
iv) Exp is repeated a large number of times say $n \longrightarrow \infty$.

HyperGeometeric Experiment:- An Exp is called HyperGeometeric Experiment if it has following properties.

1) Exp has only two outcomes, Success OR Failure
2) Probability of Success changes from trial to trial.
3) Trials are indefendent.
4) Exp is repeated a fixed no of times say n.

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2 - Bernoulli's Distribution

The Probability Dist is given by,

$$F(x) = f(x;p) = p^x (1-p)^{1-x} \quad ; x = 0,1$$

$$= p^x (1-p)^{1-x}$$

$$= p^x q^{1-x} \quad ; x = 0,1$$

$\therefore \quad p+q = 1$

also.

p = Probability of success.

q = Probability of failure.

Problem 1) Find Mean & Variance of Bernoulli's Distribution.

$$\text{Mean} = E(X) = \sum_{x=0}^{1} x f(x)$$

$$= \sum_{x=0}^{1} x \, p^x (1-p)^{1-x}$$

$$= 0 + p = p$$

$$E(X^2) = \sum_{x=0}^{1} x^2 f(x)$$

$$= \sum_{x=0}^{1} x^2 \cdot p^x (1-p)^{1-x}$$

$$= 0 + p$$

$$= p.$$

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$Var = E(X^2) - [E(X)]^2$

$= p - p^2$

$= p(1-p)$ Required.

$= pq$

Problem 2) Derive Moment Generating Function of Bernoulli's dist use it to find four moments about origin then calculate moments about Mean.

Solution:- $M_x(t) = E(e^{xt})$

$= \sum_{x=0}^{1} e^{tx} f(x)$

$= \sum_{x=0}^{1} e^{tx} p^x (1-p)^{1-x}$

$= (1-p) + e^t p$

$= q + e^t p$

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which is M.G.F of Bernoulli's Distribution.

$M_x(t) = q + e^t p$

$M_x(t) = q + p\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right]$

$= q + p + \frac{t}{1!}p + \frac{pt^2}{2!} + \frac{pt^3}{3!} + \cdots$

$= 1 + tp + p\frac{t^2}{2!} + p\frac{t^3}{3!} + \cdots$

So,

$\mu_1' = p$ (By Comparing of Co-eff $\frac{t}{1!}$)

$\mu_2' = p$ (By Comp of Co-eff $\frac{t^2}{2!}$)

$\mu_3' = p$ (By Comp of Co-eff $\frac{t^3}{3!}$)

$\mu_4' = p$ (By Comp of Co-eff $\frac{t^4}{4!}$)

Moments about Mean:-

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$\mu_1 = 0$

$\mu_2 = \mu_2' - (\mu_1')^2$

$= p - p^2$

$= p(1-p)$

$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2\mu_1'^3$

$= p - 3pp + 2p^3$

$= p - 3p^2 + 2p^3$

$\mu_3 = 2p^3 - 3p^2 + p$

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$$\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6(\mu_1')^2(\mu_2') - 3(\mu_1')^4$$

$$\mu_4 = p - 4p^2 + 6p^3 - 3p^4$$

$$\mu_4 = -3p^4 + 6p^3 - 4p^2 + p$$

Required.

3. "Binomial Distribution"

Let an Experiment is repeated n times having only two outcomes sucess & Failure. let p is Probability of success & q is Probability of Failure, then Probability of x-success is given by Binomial Distribution as,

$$f(x) = P(X=x) = \binom{n}{x} p^x q^{n-x} \quad ; x = 0,1,2,\ldots\ldots,n$$

having two Parameters p & n.
This is Generlization of Bernouli's Distribution.

Problem 1) Find Mean & Variance of Binomial Dist.

Solution > By def,

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Mean = $E(X) = \sum x f(x)$

$= \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x}$

$= \sum_{x=0}^{n} x \frac{n!}{x!(n-x)!} p^x q^{n-x}$

$= \sum_{x=0}^{n} x \frac{n!}{x(x-1)!(n-x)!} p^x q^{n-x}$

$= \sum_{x=1}^{n} \frac{n(n-1)!}{(x-1)!(n-x)!} p \cdot p^{x-1} q^{n-x}$

$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} q^{n-x}$

$= np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} q^{n-x}$

$= np \, (q+p)^{n-1}$

but $q+p = 1$ so,

$= np(1)$

$\therefore$ Mean $= np$

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$$E(X^2) = \sum x^2 f(x)$$

$$= \sum_{x=0}^{n} x^2 \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \big(x(x-1)+x\big) \cdot \frac{n!}{x!\,(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \big(x(x-1)+x\big) \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1) \binom{n}{x} p^x q^{n-x} + \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=2}^{n} \cancel{x(x-1)} \frac{n(n-1)(n-2)!}{\cancel{x(x-1)}(x-2)!\,(n-x)!} p^{x-2} p^2 q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \binom{n-2}{x-2} p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 (p+q)^{n-2} + np$$

$$E(X^2) = n^2p^2 - np^2 + np$$

Now,

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np(1-p)$$

$$Var(X) = npq \quad \because 1-p = q$$

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Problem 2) Find m.g.f of Binomial Dist use it to find Mean & Variance.

Solution:- By defination,

$$M_x(t) = E(e^{tx})$$

$$= \Sigma e^{tx} f(x)$$

$$= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} p^x e^{tx} \binom{n}{x} q^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x q^{n-x}$$

$$= (q + pe^t)^n$$

which is m.g.f for Binomial.

Now,

$$\mu_1' = \frac{d}{dt}[M_x(t)]_{t=0} = n(q+pe^t)^{n-1}(0+pe^t)$$

$$\mu_1' = n(q+p)^{n-1}(p)$$

$$\mu_1' = np(1)^{n-1}$$

$$\mu_1' = np = \text{Mean.}$$

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$\mu_2' = \frac{d^2}{dt^2}\left[M_x(t)\right]_{t=0} = n(n-1)(q+pe^t)^{n-2}$

$+ (pe^t)\, pe^t$

$+ n(q+pe^t)^{n-1} pe^t$

$\mu_2' = n(n-1)p^2 + pn$

$\mu_2' = n^2p^2 - np^2 + pn$

Now,

$Var(X) = \mu_2' - (\mu_1')^2$

$= n^2p^2 - np^2 + np - n^2p^2$

$= np(1-p)$

$Var(X) = npq \qquad \because 1-p = q$

(*) Problem 3) Prove that for Binomial Distribution

$$\mu_{r+1} = pq\left[nr\,\mu_{r-1} + \frac{d}{dp}\mu_r\right]$$

is also called Recurrance Relation.

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Solution:- The rth moment about Mean is given by,

$$\mu_r = E(X - \bar{X})^r$$

For Binomial we know

$$\bar{X} = np$$

$$\mu_r = E(X - np)^r$$

$$= \sum (x - np)^r f(x)$$

$$\mu_r = \sum_{x=0}^{n} (x - np)^r \binom{n}{x} p^x (1-p)^{n-x}$$

Diff w.r.t p

$$\frac{d\mu_r}{dp} = \sum_{x=0}^{n} \binom{n}{x} \Big\{ -nr(x-np)^{r-1} p^x (1-p)^{n-x}$$

$$+ (x-np)^r x p^{x-1} (1-p)^{n-x}$$

$$- (x-np)^r p^x (n-x)(1-p)^{n-x-1} \Big\}$$

$$\frac{d\mu_r}{dp} = -nr \sum_{x=0}^{n} \binom{n}{x} (x-np)^{r-1} p^x (1-p)^{n-x}$$

$$+ \sum_{x=0}^{n} \binom{n}{x} (x-np)^r x p^{x-1} (1-p)^{n-x}$$

$$- \sum_{x=0}^{n} \binom{n}{x} (x-np)^r p^x (n-x) (1-p)^{n-x-1}$$

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$$= -nr\mu_{r-1} + \sum_{x=0}^{n}\binom{n}{x}(x-np)^r p^x (1-p)^{n-x} \cdot \left[x/p - n-x/1-p\right]$$

$$= -nr\mu_{r-1} + \sum_{x=0}^{n}\binom{n}{x}(x-np)^r p^x (1-p)^{n-x} \cdot \left[\frac{x-pn}{p(1-p)}\right]$$

$$= -nr\mu_{r-1} + \frac{1}{pq}\sum_{x=0}^{n}\binom{n}{x}(x-np)^{r+1} p^x (1-p)^{n-x}$$

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$$= -nr\mu_{r-1} + \frac{1}{pq}\mu_{r+1}$$

$\therefore$

$$\mu_{r+1} = pq\left[nr\mu_{r-1} + \frac{d}{dp}\mu_r\right]$$

As Required.

*Derivation of Binomial Distribution

Let an experiment be repeated "n" times having only two outcomes. Let p is probability of success & q is probability of failure we further assume that

i) The possibility of success remains constant from trial to trial.

ii) Trials are independent.

let X denotes the number of success

$P[X=0] = P[F \cdot F \cdot F \cdot F \cdots\cdots F]$

$= P(F) \cdot P(F) \cdot P(F) \cdots P(F)$ $\because P(XY) = P(X) \cdot P(Y)$

$= q \cdot q \cdot q \cdots\cdots q$

$= q^n$

$P[X=1] = P[S \cdot F \cdot F \cdot F \cdots\cdots F]$

$= P(S) \cdot P(F) \cdot P(F) \cdots P(F)$

$= p\, q\, q \cdots\cdots q$

$= p q^{n-1}$

$P[X=2] = P[S \cdot S \cdot F \cdot F \cdot F \cdots\cdots F]$

$= P(S) \cdot P(S) \cdot P(F) \cdot P(F) \cdots P(F)$

$= p \cdot p \cdot q \cdot q \cdot q \cdots q$

$= p^2 q^{n-2}$

Similarly,

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$P[X=x] = P[S \cdot S \cdot S \cdots S \cdot F \cdot F \cdot F \cdots F]$

$= P(S) \cdot P(S) \cdot P(S) \cdots P(S) \cdot P(F) \cdot P(F) \cdots P(F)$

$= p^x q^{n-x}$

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$P[X=n] = P[S \cdot S \cdot S \cdots S]$

$= P(S) \cdot P(S) \cdot P(S) \cdots P(S)$

$= p^n$

Now

$P[X=x] = p^x q^{n-x}$

is the Probability of x success in some definate order. But Probability of x success in any order will be,

$P[X=x] = \binom{n}{x} p^x q^{n-x}$

which is General term in the Expansion of $(q+p)^n$.

As Required.

Problem:- Find first four moments about Mean of the Binomial Distribution. Also calculate β_1, β_2 for Binomial Distribution.

Solution:- By def

$$E(X) = \sum x f(x)$$

$$= \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=1}^{n} x \frac{n(n-1)!}{x(x-1)!(n-x)!} p^{x-1} \cdot p\, q^{n-x}$$

$$= np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} q^{n-x}$$

$$= np\,(p+q)^{n-1}$$

As $p+q = 1$ so,

$$E(X) = np$$

$$\Rightarrow \mu_1' = np$$

$$E(X^2) = \sum x^2 f(x) \Rightarrow \sum [x(x-1)+x] f(x)$$

$$= \sum x(x-1) f(x) + \sum x f(x)$$

$$= \sum_{x=0}^{n} x(x-1) \binom{n}{x} p^x q^{n-x} + np$$

$$= \sum_{x=2}^{n} x(x-1) \frac{n(n-1)(n-2)!}{x(x-1)(x-2)!(n-x)!} p^{x-2} p^2 q^{n-x} + np$$

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$$E(X^2) = n(n-1)p^2 \sum_{x=2}^{n} \binom{n-2}{x-2} p^{x-2} q^{n-x} + np$$

$$E(X^2) = n(n-1)p^2 (p+q)^{n-2} + np$$

$$\mu_2' = n(n-1)p^2 + np$$

$$E(X^3) = \sum x^3 f(x)$$

$$= \sum [x(x-1)(x-2) + 3x(x-1) + x] f(x)$$

$$E(X^3) = \sum x(x-1)(x-2) f(x) + 3 \sum x(x-1) f(x) + \sum x f(x)$$

$$E(X^3) = \sum_{x=0}^{n} x(x-1)(x-2) \binom{n}{x} p^x q^{n-x} + 3[n(n-1)p^2 + \ldots] + np$$

$$E(X^3) = \sum_{x=3}^{n} x(x-1)(x-2) \frac{n(n-1)(n-2)(n-3)!}{x(x-1)(x-2)(x-3)!(n-x)!} p^{x-3} p^3 q^{n-x} + 3n(n-1)p^2 + np$$

$$E(X^3) = n(n-1)(n-2)p^3 \sum_{x=3}^{n} \binom{n-3}{x-3} p^{x-3} q^{n-x} + 3n(n-1)p^2 + np$$

$$\mu_3' = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$$

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$E(x^4) = \sum x^4 f(x)$

$= \sum [x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x] f(x)$

$= \sum x(x-1)(x-2)(x-3) f(x) + 6 \sum x(x-1)(x-2) f(x) + 7 \sum x(x-1) f(x) + \sum x f(x)$

$= \sum_{x=0}^{n} x(x-1)(x-2)(x-3) \binom{n}{x} p^x q^{n-x}$

$+ 6(n(n-1)(n-2)p^3)$
$+ 7(n(n-1)p^2) + np$

$= \sum_{x=4}^{n} x(x-1)(x-2)(x-3) \frac{n(n-1)(n-2)(n-3)(n-4)!}{x(x-1)(x-2)(x-3)(x-4)!(n-x)!} p^{x-4} p^4 q^{n-x}$

$+ 6n(n-1)(n-2)p^3 +$
$(7n(n-1)p^2) + np$

$= n(n-1)(n-2)(n-3)p^4 \sum_{x=4}^{n} \binom{n-4}{x-4} p^{x-4} q^{n-x}$

$+ 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$

$\therefore$

$\mu_4' = n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$

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Moments about Mean,

$\mu_1 = 0$

$\mu_2 = \mu_2' - (\mu_1')^2$

$\mu_2 = n(n-1)p^2 + np - n^2p^2$

$= n^2p^2 - np^2 + np - n^2p^2$

$= np(1-p)$

$\mu_2 = npq$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

$\mu_3 = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + 4np$
$\quad - 3(n(n-1)p^2 + np)(np) + 2(np)^3$

$\mu_3 = n(n-1)\left[(n-2)p^3 + 3p^2\right]$
$\quad + np\left[4 - 3n(n-1)p^2 - 3np + 2n^2p^2\right]$

$\mu_3 = n(n-1)\left[np^3 - 2p^3 + 3p^2\right]$
$\quad + np\left[4 - 3n^2p^2 + 3np^2 - 3np + 2n^2p^2\right]$

$\mu_3 = n^3p^3 - n^2p^3 - 2n^2p^3 + 2np^3 + 3n^2p^2 - 3np^2$
$\quad + 4np - n^3p^3 + 3n^2p^3 - 3n^2p^2$ ...

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$\mu_3 = 2np^3 - 3np^2 + 4np$

$\mu_3 = np[2p^2 - 3p + 4]$

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$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$\mu_4 = n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3$
$+ 25n(n-1)p^2 + 32np - 4np[n(n-1)(n-2)p^3$
$+ 3n(n-1)p^2 + 4np]$

$+ 6[n(n-1)p^2 + np]n^2p^2 - 3n^4p^4$

$\mu_4 = n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3$
$+ 25n(n-1)p^2 + 32np - 4n^2(n-1)(n-2)p^4$
$- 12n^2(n-1)p^3 - 16n^2p^2 + 6n^3(n-1)p^4 + 6n^3p^3$
$- 3n^4p^4$

$\mu_4 = n(n-1)(n-2)p^3[(n-3)p + 6 - 4np]$
$+ n(n-1)p^2[25 - 12np + 6n^2p^2] + 32np$
$+ 6n^3p^3 - 3n^4p^4 - 16n^2p^2$

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Moments About Mean

$\mu_1 = 0$

$\mu_2 = \mu_2' - (\mu_1')^2$

$= n(n-1)p^2 + np - (np)^2$

$= n^2p^2 - np^2 + np - (np)^2$

$= np(1-p)$

$\mu_2 = npq$

$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2\mu_1'^3$

$\mu_3 = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$
$- 3np\{n(n-1)p^2 + np\} + 2(np)^3$

$= n(n-1)p^2[(n-2)p + 3 - 3pn\} + 2n^3p^3 + np$
$- 3n^2p^2$

$= (n^2p^2 - np^2)(-2np - 2p + 3) - 3n^2p^2 + 2n^3p^3 + np$

$= -2n^3p^3 - 2n^2p^3 + 3n^2p^2 + 2n^2p^3 + 2np^3 - 3np^2$
$- 3n^2p^2 + 2n^3p^3 + np$

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$= 2np^3 - 3n^2p^2 + np$

$= np\left[2p^2 - 3p + 1\right]$

$= np\left[2p^2 - 2p - p + 1\right]$

$= np\left\{2p(-1+p) - 1(-1+p)\right\}$

$= np\left[(-1+p)(2p-1)\right]$

$= np\,(1-2p)(-p+1)$

$= np(1-p)(1-p-p)$

$\mu_3 = npq(q-p)$

$\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6(\mu_1')^2\mu_2' - 3\mu_1'^4$

$= \{n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3$
$+ 7n(n-1)p^2 + np\}$

$- 4\{(np)\left[n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np\right]\}$

$+ 6\{n^2p^2(n(n-1)p^2 + np)\} - 3n^4p^4$

$= n(n-1)(n-2)p^3\left[np - 3p + 6 - 4np\right] - 3n^4p^4$

$+ n(n-1)p^2\left[7 - 12np + 6n^2p^2\right] + np - 4n^2p^2 + 6n^3p^3$

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$$= n(n-1)p^2\left[(np-2p)(-3np-3p+6)+7-12np+6n^2p^2\right]$$

$$-3n^4p^4+6n^3p^3-4n^2p^2+np$$

$$= (n^2p^2-np^2)\left[-3n^2p^2-3np^2+6np+6np^2+6p^2-12p+7-12np+6n^2p^2\right]$$

$$-3n^4p^4+6n^3p^3-4n^2p^2+np$$

$$= (n^2p^2-np^2)\left[3n^2p^2+3np^2-6np+6p^2-12p+7\right]$$

$$-3n^4p^4+6n^3p^3-4n^2p^2+np$$

$$= 3n^4p^4+3n^3p^4-6n^3p^3+6n^2p^4-12n^2p^3+7n^2p^2$$
$$-3n^3p^4-3n^2p^4+6n^2p^3-6np^4+12np^3-7np^2$$
$$-3n^4p^4+6n^3p^3-4n^2p^2+np$$

$$= 3n^2p^4-6n^2p^3+3n^2p^2-6np^4+12np^3-7np^2+np$$

$$= np\left[3np^3-6np^2+3np-6p^3+12p^2-7p+1\right]$$

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$$= np\left[3np(1+p^2) - 6np^2 - 6p^2(2-p) - 7p+1\right]$$

$$= np\left[3np(1+p^2-2p) - 6p^2(1+1-p) - 7p+1\right]$$

$$= np\left[3np(1-p)^2 - 6p^2(1+q) - 7p+1\right]$$

$$= np\left[3npq^2 - 6p^2 - 6p^2q - 7p+1\right]$$

$$\mu_4 = np\left[3(n-2)pq + 1\right]$$

Now we have to find β- Coefficients,

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3n^2p^2q^2 + npq(1-6pq)}{(npq)^2}$$

$$\frac{3npq + 1 - 6pq}{npq}$$

$$= 3 + \left(\frac{1}{npq} - \frac{6}{n}\right)$$

As Required.

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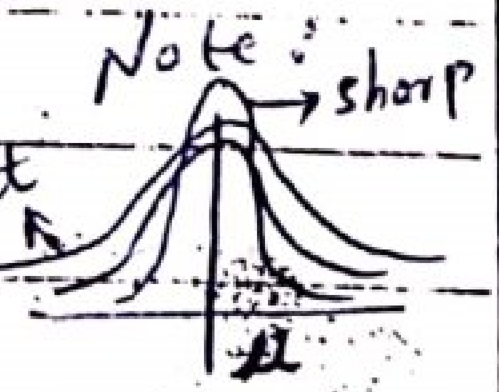
$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{n^2p^2q^2(q-p)^2}{n^3p^3q^3}$$

$$\beta_1 = \frac{(q-p)^2}{npq}$$ As Required.

Note:



β_1 determins the sharpness of curve while β_2 determins the skewness of the curve

For symmetric graph $\beta_2 = 3$ For +vely skewed $\beta_2 > 3$ For -vely skewed $\beta_2 < 3$

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*Question:- Find Mean & Var (S.D) of Binomial Distribution $(q+p)^3$.

Solution $n = 3$

x	$f(x) = \binom{3}{x} p^x q^{3-x}$	$xf(x)$	$x^2 f(x)$
0	$\binom{3}{0} p^0 q^{3-0} = q^3$	0	0
1	$\binom{3}{1} p^1 q^{3-1} = 3pq^2$	$3pq^2$	$3pq^2$
2	$\binom{3}{2} p^2 q^{3-2} = 3p^2q$	$6p^2q$	$12p^2q$
3	$\binom{3}{3} p^3 q^{3-3} = p^3$	$3p^3$	$9p^3$

Now,

$$\text{Mean} = E(X) = \sum x f(x)$$

$$= 3pq^2 + 6p^2q + 3p^3$$

$$= 3p(q^2 + 2pq + p^2)$$

$$= 3p(q+p)^2$$

$$E(x) = 3p$$

$$E(x^2) = \sum x^2 f(x)$$

$$= 3pq^2 + 12p^2q + 9p^3$$

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$= 3p\,(q^2 + 4pq + 3p^2)$

$= 3p\,(q^2 + 2pq + 2pq + p^2 + 2p^2)$

$= 3p\,((q+p)^2 + 2pq + 2p^2)$

$= 3p\,[1 + 2p(p+q)]$

$E(x^2) = 3p\,[1 + 2p]$

$Var(x) = E(x^2) - (E(x))^2$

$= 3p[1+2p] - 9p^2$

$= 3p + 6p^2 - 9p^2$

$= 3p - 3p^2$

$= 3p(1-p)$

$= 3pq$

$S.D(x) = \sqrt{Var(x)}$

$\therefore S.D(x) = \sqrt{3pq}$ As Required.

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Problem:- In a Binomial Dist the Mean & S.D were found to be 36 & 4.8 respectively. Find p, q, n.

Solution:- Given

Mean = 36

$\Rightarrow \quad np = 36 \rightarrow (1)$

S.D = 4.8

$\sqrt{npq} = 4.8$

$npq = (4.8)^2$

$npq = 23.04$

using (1) we have

$36q = 23.04$

$q = \frac{23.04}{36} = 0.64$ Required.

As $p+q = 1$

$p = 1-q$

$p = 1 - 0.64 = 0.36$ Required.

$\therefore \quad n = \frac{36}{p} = \frac{36}{0.36}$

$n = 100$ Required.

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Problem:- A r.v is binomially distributed with Mean 3 & Var 2. Compute probability $P[X=7]$

Solution:-

$$\text{Mean} = 3$$
$$np = 3$$
$$\text{Var} = 2$$
$$npq = 2$$
$$3q = 2$$
$$q = 2/3$$

$$p = 1 - q$$
$$p = 1 - 2/3 = 1/3$$

$$n.p = 3$$
$$n = 3/p = 9$$

$$\therefore P[X=7] = \binom{n}{7} p^7 q^{n-7}$$
$$= \binom{9}{7} \left(\frac{1}{3}\right)^7 \left(\frac{2}{3}\right)^{9-7}$$
$$= 36 \cdot \frac{1}{3^7} \cdot \frac{4}{9}$$
$$= \frac{16}{3^7} \quad \text{Required.}$$

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Problem:- Using Binomial Distribution find the Probability of

i) 3 success in 8 trial when $P = 0.4$

ii) 2 or fewer success in 9 trials when $P = 0.4$

i) $n = 8, \quad P = 0.4, \quad q = 1 - P$

$q = 0.6$

$x = 3$

$$P[x = 3] = \binom{8}{3} (0.4)^3 (0.6)^{8-3}$$

$$= 56 \times 0.064 \times 0.07776$$

$$= 0.278$$

ii) $P[x \leq 2] = ?$

$n = 9, \quad x = 0, 1, 2, \quad P = 0.4, \quad q = 0.6$

$$P[x \leq 2] = P[x = 0] + P[x = 1] + P[x = 2]$$

$$= \binom{9}{0} (0.4)^0 (0.6)^{9-0} + \binom{9}{1} (0.4)^1 (0.6)^{9-1}$$

$$+ \binom{9}{2} (0.4)^2 (0.6)^{9-2}$$

$$= 0.010 + 0.060 + 0.161$$

$$= 0.231$$ Required

Problem:- Let X be Binomial R.V & Moment generating function is given by,

$$M_0(t) = \left(\frac{1}{4} + \frac{3}{4}e^t\right)^{12}$$

Find $E(x)$, $Var(x)$ & $P[x \geq 10]$

Solution:- In case of Binomial Dist the moment generating function is,

$$M_0(t) = (q + pe^t)^n$$

By Comparing we get

$$q = \frac{1}{4}, \quad p = 3/4, \quad n = 12$$

i) $E(x) = n.p$

$$= 12 \times 3/4 = 9$$

ii) $Var(x) = npq$

$$= 12 \times 3/4 \times 1/4$$

$$= 9/4$$

$$= 2.25$$

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(ii) $P[X \geq 10] = P[X=10] + P[X=11] + P[X=12]$

$= \binom{12}{10} p^{10} q^{12-10} + \binom{12}{11} p^{11} q^{12-11} + \binom{12}{12} p^{12} q^{12-12}$

$= \binom{12}{10}\left(\frac{3}{4}\right)^{10}\left(\frac{1}{4}\right)^{2} + \binom{12}{11}\left(\frac{3}{4}\right)^{11}\left(\frac{1}{4}\right)^{1} + \binom{12}{12}\left(\frac{3}{4}\right)^{12}\left(\frac{1}{4}\right)^{0}$

$= \frac{66}{16}\left(\frac{3}{4}\right)^{10} + \frac{12}{4}\left(\frac{3}{4}\right)^{11} + \left(\frac{3}{4}\right)^{12}$

$= \frac{33}{8}\left(\frac{3}{4}\right)^{10} + 3\left(\frac{3}{4}\right)^{11} + \left(\frac{3}{4}\right)^{12}$

$= 0.232 + 0.126 + 0.031$

$= 0.390$ As Required.

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Covariance of Two Random Variables

The Covariance of two r.v X & Y is a numerical value to which their values tend to increase or decrease together. It is denoted by Cov(X,Y) or σ_{xy} & is defined as, "The Expected value of the product $[X-E(X)][Y-E(Y)]$"

$$\text{COV}(X,Y) = E\{[X-E(X)][Y-E(Y)]\}$$

$$= E\{XY - XE(Y) - YE(X) + E(X)E(Y)\}$$

$$= E(XY) - E(Y)E(X) - E(X)E(Y) + E(X)E(Y)$$

$$= E(XY) - E(X)E(Y)$$

If X & Y independent so then

$$\text{Cov}(X,Y) = \sigma_{XY} = 0$$

Problem:- Show that for Binomial Variate

$$\text{COV}\left(\frac{X}{n}, \frac{n-X}{n}\right) = \frac{-pq}{n}$$

Solution:- $\text{COV}(X,Y) = E(XY) - E(X)E(Y)$

$$= E\left(\frac{X}{n} \cdot \frac{n-X}{n}\right) - E\left(\frac{X}{n}\right) E\left(\frac{n-X}{n}\right)$$

$$= E\left(\frac{X}{n} \cdot \left(1-\frac{X}{n}\right)\right) - \frac{1}{n}E(X)\, E(1 - X/n)$$

$$= E\left(\frac{X}{n} - \frac{X^2}{n^2}\right) - \frac{1}{n}E(X)\left[1 - \frac{1}{n}E(X)\right]$$

$$= \frac{1}{n}E(X) - \frac{1}{n^2}E(X^2) - \frac{1}{n}E(X)\left[1 - \frac{1}{n}E(X)\right]$$

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As we Know,

$$Var(X) = E(X^2) - [E(X)]^2$$

$$npq = E(X^2) - (np)^2$$

$$\therefore E(X^2) = npq + n^2p^2$$

$$= np(q + np)$$

$\because$

$$= \frac{1}{n}(np) - \frac{1}{n^2}(np(q+np)) - \frac{1}{n}(np)\left[1 - \frac{1}{n} \cdot np\right]$$

$$= p - \frac{pq}{n} - p^2 - p[1-p]$$

$$= \cancel{p} - \frac{pq}{n} - \cancel{p^2} - \cancel{p} + \cancel{p^2}$$

$$= \frac{-pq}{n}$$ As Required.

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MODE OF Binomial Distribution

To find Mode of Binomial Dist

$\Rightarrow$ To find Variate of Max Frequency

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Suppose that Greatest term in Binomial Distribution $(q+p)^n$ is "r" s.t

$$P[X=r-1] \leq P[X=r] \geq P[X=r+1]$$

$$\binom{n}{r-1} p^{r-1} q^{n-r+1} \leq \binom{n}{r} p^{r} q^{n-r} \geq \binom{n}{r+1} p^{r+1} q^{n-r-1}$$

Dividing by $p^r q^{n-r}$

$$\binom{n}{r-1} q \cdot \frac{1}{p} \leq \binom{n}{r} \geq \binom{n}{r+1} p \cdot \frac{1}{q}$$

$$\frac{n!}{(r-1)!\,(n-r+1)!} \frac{q}{p} \leq \frac{n!}{r!\,(n-r)!} \geq \frac{n!}{(r+1)!\,(n-r-1)!} \frac{p}{q}$$

$$\frac{r!}{(r-1)!\,(n-r+1)!} \frac{q}{p} \leq \frac{1}{(n-r)!} \geq \frac{r!}{(r+1)!\,(n-r-1)!} \frac{p}{q}$$

$$\frac{r(r-1)!}{(r-1)!\,(n-r+1)!} \frac{q}{p} \leq \frac{1}{(n-r)!} \geq \frac{r!}{(r+1)\,r!\,(n-r-1)!} \frac{p}{q}$$

$$\frac{r}{(n-r+1)!} \frac{q}{p} \leq \frac{1}{(n-r)!} \geq \frac{1}{(r+1)(n-r-1)!} \frac{p}{q}$$

$$\frac{r\,(n-r)!}{(n-r+1)!} \frac{q}{p} \leq 1 \geq \frac{(n-r)!}{(r+1)\,(n-r-1)!} \frac{p}{q}$$

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$$\frac{r(n-r)!}{(n-r+1)(n-r)!}\cdot\frac{q}{p} \le 1 \ge \frac{(n-r)(n-r-1)!}{(r+1)(n-r-1)!}\frac{p}{q}$$

$$\frac{r}{n-r+1}\frac{q}{p} \le 1 \ge \frac{n-r}{r+1}\frac{p}{q}$$

Now,

$$\frac{r}{n-r+1}\frac{q}{p} \le 1$$

$$r \le \frac{p(n-r+1)}{q}$$

$$qr \le pn - pr + p$$

$$pr + qr \le pn + p$$

$$r \le pn + p$$

$$r \le p(n+1)$$

Now,

$$\frac{n-r}{r+1}\frac{p}{q} \le 1$$

$$pn - pr \le qr + q$$

$$pn - q \le r$$

$$\Rightarrow r \ge pn - q$$

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$np - q \leq r \leq pn + p$

$np - q \leq r \leq np + p$

→ Mode of Binomial Dist. is an integer lying b/w $np-q$ & $np+p$ which is np.

Hence

$$Mode = r = np$$

As Required.

*Cumulant Generating Function

The Cumulant Generating Function of a Random Variable is defined as,

$$K(t) = \log_e M_0(t)$$

$$= K_1 t + K_2 \frac{t^2}{2!} + K_3 \frac{t^3}{3!} + \dots + K_r \frac{t^r}{r!} + \dots$$

The Co-efficients $K_1, K_2, K_3, \dots$ are first, second & third Cumulants & $K(t)$ is called Cumulant Generating function.

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Question:- Find Cumulant function of Binomial Distribution & have obtain its 1st four Cumulants.

Solution:- As we know that

$$K(t) = \log_e M_o(t)$$

where $M_o(t) = (q + pe^t)^n$

$$= (1 - p + pe^t)^n \quad \because q = 1 - p$$

$$M_o(t) = (1 + p(e^t - 1))^n$$

$$K(t) = \log_e [1 + p(e^t - 1)]^n$$

$$= n \log_e [1 + p(e^t - 1)]$$

$$= n \log_e \left[1 + p\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots - 1\right)\right]$$

$$= n \log_e \left[1 + pt + p\frac{t^2}{2!} + p\frac{t^3}{3!} + p\frac{t^4}{4!} + \dots\right]$$

$$= n \log_e \left[1 + p\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots\right)\right]$$

Since

$$\log(1 + Z) = Z - \frac{Z^2}{2} + \frac{Z^3}{3} - \frac{Z^4}{4} + \dots$$

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$$K(t) = n\Big[p\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\right) - \frac{1}{2}\left[p\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\right)\right]^2 + \frac{1}{3}\left[p\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\right)\right]^3 + \frac{1}{4}\left[p\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\right)\right]^4+\cdots\Big]$$

$$K(t) = n\Big[pt + \frac{1}{2}pt^2 + \frac{1}{6}pt^3 + \frac{1}{24}pt^4 - \frac{1}{2}p^2t^2 - \frac{1}{8}p^2t^4 - \frac{1}{2}p^2t^3 - \frac{1}{6}p^2t^4 + \frac{1}{3}p^3t^3 + \frac{1}{2}p^3t^4 - \frac{1}{4}p^4t^4 + \cdots\Big]$$

$$K(t) = n\Big[pt + (p-p^2)\frac{t^2}{2!} + (p-3p^2+2p^3)\frac{t^3}{3!} + (p-3p^2-4p^2+12p^3-6p^4)\frac{t^4}{4!}+\cdots\Big]$$

Now Compare the Co-efficients of $\frac{t^r}{r!}$ we get

$K(t_1) = np$

$K(t_2) = n(p-p^2) = np(1-p) = npq$

$K(t_3) = n(p-3p^2+2p^3) = npq(q-p)$

$K(t_4) = n(p-3p^2-4p^2+12p^3-6p^4) = npq(1-6pq)$

As Required.

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Relation b/w Cumulants & Moments

By def we know

$$K(t) = \log_e M_o(t)$$

$$K_1(t) + K_2\left(\frac{t^2}{2!}\right) + K_3\left(\frac{t^3}{3!}\right) + K_4\left(\frac{t^4}{4!}\right) + \cdots + K_r\frac{t^r}{r!} + \cdots$$

$$= \log_e\left[\sum_{r=0}^{\infty} \mu_r' \frac{t^r}{r!}\right]$$

$$= \log_e\left[1 + \sum_{r=1}^{\infty} \mu_r' \frac{t^r}{r!}\right]$$

$$= \log_e(1+Z) \quad \text{where } Z = \sum_{r=1}^{\infty} \mu_r' \frac{t^r}{r!}$$

$$= \left(\sum_{r=1}^{\infty} \mu_r' \frac{t^r}{r!}\right) - \frac{1}{2}\left(\sum_{r=1}^{\infty} \mu_r' \frac{t^r}{r!}\right)^2 +$$

$$\frac{1}{3}\left(\sum_{r=1}^{\infty} \mu_r' \frac{t^r}{r!}\right)^3 + \cdots$$

$$= \left[t\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \frac{t^4}{4!}\mu_4' + \cdots\right]$$

$$-\frac{1}{2}\left[\mu_1' t + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \cdots\right]^2$$

$$+\frac{1}{3}\left[\mu_1' t + \mu_2' \frac{t^2}{2!} + \cdots\right]^3$$

$$-\frac{1}{4}\left[\mu_1' t + \cdots\right]^4 + \cdots$$

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$= \mu_1'(t) + (\mu_2' - \mu_1'^2)\frac{t^2}{2!} + (\mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3)\frac{t^3}{3!}$

$+ (\mu_4' - 4\mu_3'\mu_1' - 3\mu_2'^2 + 12\mu_2'\mu_1'^2 - 6\mu_1'^2)\frac{t^4}{4!}$

Now by Comparing Co-efficent we get

Note: $K_1 = \mu_1'$, $K_2 = \mu_2$, $K_3 = \mu_3$
$K_4 = \mu_4 - 3\mu_2^2$

$\therefore K_1 = \mu_1'$

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$K_2 = \mu_2' - \mu_1'^2$

$K_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

$K_4 = \mu_4' - 4\mu_3'\mu_1' - 3\mu_2'^2 + 12\mu_2'\mu_1'^2 - 6\mu_1'^2$

which is Required Relation b/w comulants & Moments.

Reproductive Property of Binomial Dist

Show that distribution of sum of two independent random Binomial variates X & Y with Parameters n_1, p & n_2, p respectively is also a Binomial Distribution with Parameter $(n_1+n_2), p$.

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Proof:- let x & y be two independent random variables with parameters n_1, p & n_2, p then

$$M_x(t) = (q + pe^t)^{n_1}$$

$$M_y(t) = (q + pe^t)^{n_2}$$

As we know that

$$M_{x+y} = M_x(t) \cdot M_y(t)$$

$$= (q + pe^t)^{n_1} (q + pe^t)^{n_2}$$

$$= (q + pe^t)^{n_1 + n_2}$$

As Required.

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4:- Poisson Distribution

when number of Trials "n" in the Binomial Dist becomes very Large & Probability of success p becomes very small but np remains constant then Binomial Distribution approaches in limit to another distribution which is called "Poisson Distribution" due to french Mathematician S. D. Poisson.

The Poisson Dist is given by,

$$P[X=x] = \frac{m^x e^{-m}}{x!} \quad ; x = 0, 1, 2, \ldots \infty$$

Here m is the only Parameter of the distribution.

Question:- Derive Poisson Distribution as limiting case of Binomial Dist.

OR

Poisson Probability Law is limiting form of the Binomial Probability Law when number of trials "n" is very very Large & Probability of Success "p" is very small

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Such that the product "np" remains finite say m.

OR

Derive Poisson Distribution.

Proof:- The Binomial Probability laws is given by,

$P[X=x] = b(x; n, p) = \binom{n}{x} p^x q^{n-x}$; $x = 0, 1, 2, \ldots n$

we have to consider it under the limiting case when

i) n is indefinatley large i.e, $n \to \infty$.

ii) p is very very small i.e, $p \to 0$

iii) $np = m$

$p = m/n$

$q = 1 - m/n$

$\because \quad q = 1 - p$

So, we have

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$$P[X=x] = \frac{n!}{x!\,(n-x)!}\left(\frac{m}{n}\right)^{x}\left(1-\frac{m}{n}\right)^{n-x}$$

$$= \frac{n(n-1)(n-2)\cdots(n-x+1)(n-x)!}{x!\,(n-x)!}\left(\frac{m}{n}\right)^{x}\left(1-\frac{m}{n}\right)^{n-x}$$

$$= \frac{n(n-1)(n-2)\cdots(n-x+1)}{x!}\,\frac{m^{x}}{n^{x}}\left(1-\frac{m}{n}\right)^{-x}\left(1-\frac{m}{n}\right)^{n}$$

$$= \frac{m^{x}}{x!}\left[\frac{n(n-1)(n-2)\cdots(n-x+1)}{n.n.n\cdots n}\right]\left[1-\frac{m}{n}\right]^{-x}\left[\left(1-\frac{m}{n}\right)^{n/m}\right]^{m}$$

$$= \frac{m^{x}}{x!}\left\{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1+\frac{1-x}{n}\right)\right\}\left[1-\frac{m}{n}\right]^{-x}\left[\left(1-\frac{m}{n}\right)^{n/m}\right]^{m}$$

$$\lim_{n\to\infty} P[X=x] = \frac{m^{x}}{x!}\lim_{n\to\infty}\left[\left(1-\frac{m}{n}\right)^{-n/m}\right]^{-m}$$

$$= \frac{m^{x}}{x!}\,e^{-m}$$

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So,

$$P[X=x] = \frac{m^{x}}{x!}\,e^{-m} \quad \forall\, x = 0, 1, 2, 3, \ldots \infty$$

which is Expression for Poisson Distribution.

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Question:- Find Mean & Var of Poisson Distribution.

Solution:- As we know

$$\text{Mean} = E(X) = \sum x\, f(x)$$

$$= \sum_{x=0}^{\infty} x \frac{m^x e^{-m}}{x!}$$

$$= \sum_{x=1}^{\infty} x \frac{m^{x-1} \cdot m \cdot e^{-m}}{x(x-1)!}$$

$$= m \sum_{x=1}^{\infty} \frac{m^{x-1} e^{-m}}{(x-1)!}$$

$$= m e^{-m} \sum_{x=1}^{\infty} \frac{m^{x-1}}{(x-1)!}$$

$$= m e^{-m} \left[1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots \right]$$

$$= m e^{-m} e^{m}$$

$$= m$$

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$E(X^2) = \sum x^2 f(x)$

$= \sum_{x=0}^{\infty} x^2 \frac{m^x e^{-m}}{x!}$

$= \sum_{x=0}^{\infty} x(x-1)+x \cdot \frac{m^x e^{-m}}{x!}$

$= \sum_{x=2}^{\infty} \cancel{x(x-1)} \frac{m^x e^{-m}}{\cancel{x(x-1)}(x-2)!} + \sum_{x=0}^{\infty} x \frac{m^x e^{-m}}{x!}$

$= e^{-m} m^2 \sum_{x=2}^{\infty} \frac{m^{x-2}}{(x-2)!} + m$

$= m^2 e^{-m} \left[1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots \right] + m$

$= m^2 e^{-m} e^{m} + m \Rightarrow m^2 + m$

Now,

$Var(X) = E(X^2) - (E(X))^2$

$= \cancel{m^2} + m - \cancel{m^2}$

$Var(X) = m$

As Required.

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Question:- Prove that Poisson Dist

$$\mu_{r+1} = \lambda\left[r\mu_{r-1} + \frac{\partial \mu_r}{\partial \lambda}\right]$$

Solution:- As we know that

$$\mu_r = E(X-\bar{X})^r$$

$$\mu_r = \sum (x-\bar{x})^r f(x)$$

$$\mu_r = \sum_{x=0}^{\infty} (x-\bar{x})^r \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= \sum_{x=0}^{\infty} (x-\lambda)^r \frac{\lambda^x e^{-\lambda}}{x!} \quad \because \lambda = m$$

$$\frac{\partial \mu_r}{\partial \lambda} = \sum_{x=0}^{\infty} r(x-\lambda)^{r-1}(-1)\frac{e^{-\lambda}\lambda^x}{x!}$$

$$+ \sum_{x=0}^{\infty} (x-\lambda)^r \frac{e^{-\lambda}(-1)\lambda^x}{x!}$$

$$+ \sum_{x=0}^{\infty} (x-\lambda)^r \frac{e^{-\lambda}\, x\lambda^{x-1}}{x!}$$

$$= -r\sum_{x=0}^{\infty} (x-\lambda)^{r-1} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$+ \sum_{x=0}^{\infty} (x-\lambda)^r (-1)\frac{\lambda^x e^{-\lambda}}{x!} + \sum_{x=0}^{\infty} (x-\lambda)^r \frac{e^{-\lambda}\, x\lambda^{x-1}}{x!}$$

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$$= -r\mu_{r-1} + \sum_{x=0}^{\infty} \frac{(x-\lambda)^r e^{-\lambda} \lambda^x}{x!} \left\{-1 + x/\lambda\right\}$$

$$= -r\mu_{r-1} + \sum_{x=0}^{\infty} \frac{(x-\lambda)^r e^{-\lambda} \lambda^x}{x!} \left[\frac{x-\lambda}{\lambda}\right]$$

$$= -r\mu_{r-1} + \sum_{x=0}^{\infty} (x-\lambda)^{r+1} \frac{e^{-\lambda} \lambda^x}{x!} \frac{1}{\lambda}$$

$$= -r\mu_{r-1} + \frac{\mu_{r+1}}{\lambda}$$

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$$\lambda \frac{\partial \mu_r}{\partial \lambda} = -r\lambda \mu_{r-1} + \mu_{r+1}$$

$$\Rightarrow \quad \mu_{r+1} = \lambda \left[r\mu_{r-1} + \frac{\partial \mu_r}{\partial \lambda} \right]$$

As Required.

Problem:- Find m.g.f of Poisson Distribution & use it to find Mean & Var of the Poisson Distribution.

Solution:- By def

$$M_x(t) = E(e^{tx})$$

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$$= \sum e^{tx} f(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-m} m^x}{x!}$$

$$= e^{-m} \sum_{x=0}^{\infty} \frac{(me^t)^x}{x!}$$

$$= e^{-m}\left[1 + \frac{me^t}{1!} + \frac{(me^t)^2}{2!} + \frac{(me^t)^3}{3!} + \cdots\right]$$

$$= e^{-m} e^{me^t}$$

$$= e^{me^t - m}$$

$$M_x(t) = e^{m(e^t - 1)}$$

which is m.g.f of Poisson Distribution.

$$\frac{d}{dt}(M_x(t)) = e^{m(e^t - 1)} me^t$$

$$\mu_1' = \frac{d}{dt}(M_x(t))\Big|_{t=0} = e^{m(1-1)} me^0$$

$$\mu_1' = m$$

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$\therefore$ Mean $= \mu_1' = m$

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$$\frac{d^2}{dt^2} M_x(t) = e^{m(e^t-1)} m^2 e^{2t} + e^{m(e^t-1)} m e^t$$

$$\mu_2' = \frac{d^2}{dt^2} M_x(t) \Big|_{t=0} = m^2 + m$$

$\therefore$

$$\mu_2' = m^2 + m$$

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Now $\quad Var(x) = \mu_2' - (\mu_1')^2$

$$= m^2 + m - m^2$$

$$= m. \quad \text{As Required.}$$

Note:- Mean & Variance of Poisson Dist are Coincide.

Reproductive Property of Poisson Variates:-

If X_1 & X_2 are two independent poisson variates with Parameters m_1 & m_2 respectively, show that X_1+X_2 is also a poisson Variate with Parameter m_1+m_2.

Proof:- As X_1 & X_2 are two

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independent Poisson Variates with Parameters m_1 & m_2 then

$$M_{X_1}(t) = e^{m_1(e^t-1)}$$

$$M_{X_2}(t) = e^{m_2(e^t-1)}$$

$$M_{(X_1+X_2)}(t) = M_{X_1}(t) \cdot M_{X_2}(t)$$

$$= e^{m_1(e^t-1)} \cdot e^{m_2(e^t-1)}$$

$$= e^{(m_1+m_2)(e^t-1)}$$

Hence Proved that variate of Poisson Variate is also Poisson Variate.

Question:- Find Mode of Poisson Distb.

Solution:- let "x" be the Mode of Poisson Distribution.

$$P[X=x-1] \le P[X=x] \geq P[X=x+1]$$

$$\frac{e^{-m} m^{x-1}}{(x-1)!} \le \frac{e^{-m} m^{x}}{x!} \geq \frac{e^{-m} m^{x+1}}{(x+1)!}$$

$$\frac{m^{x-1}}{(x-1)!} \le \frac{m^{x}}{x!} \geq \frac{m^{x+1}}{(x+1)!}$$

$$\frac{m^{-1}}{(x-1)!} \leq \frac{1}{x!} \geq \frac{m}{(x+1)!}$$

$$\frac{m^{-1}\, x!}{(x-1)!} \leq 1 \geq \frac{m\, x!}{(x+1)!}$$

$$\frac{m^{-1}\, x(x-1)!}{(x-1)!} \leq 1 \geq \frac{m x!}{(x+1)\, x!}$$

$$m^{-1} x \leq 1 \geq \frac{m}{x+1}$$

$$\frac{x}{m} \leq 1 \geq \frac{m}{x+1}$$

Now Consider $\frac{x}{m} \leq 1$

$$x \leq m$$

Now Consider

$$\frac{m}{x+1} \leq 1$$

$$m \leq x+1$$

$$x \geq m-1$$

$$m-1 \leq x \leq m$$

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Hence if m is an integral value lying b/w m-1 & m then the probability of Poisson variates taking the values "m-1" & m is the "m", So, Either of them is the Mode So

Mode = M Required.

Cumulant Generating Function

Find Cumulant Generating Function of Poisson Dist & then show that cumulants of Poisson Dist are all Equal.

Proof:- By def we have

$$K(t) = \log_e(M_x(t))$$

In case of Poisson Dist $M_x(t) = e^{m(e^t-1)}$

$$K(t) = \log_e[e^{m(e^t-1)}]$$

$$K(t) = m(e^t - 1)$$

$$K(t) = m\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots - 1\right]$$

$$K(t) = m\left[t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right]$$

$k_1(t) + k_2\left(\frac{t^2}{2!}\right) + k_3\left(\frac{t^3}{3!}\right) + \cdots$

$= mt + m\left(\frac{t^2}{2!}\right) + m\left(\frac{t^3}{3!}\right) + \cdots$

Combining both sides we get

$k_1 = m$

$k_2 = m$

$k_3 = m$

$\vdots \quad \vdots$

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Hence In general all the Cumulants of Generating function are Equal.

Problem:- Find β-Coefficients for Poisson Distribution.

Solution:- Now,

$\mu_1' = E(X) = \sum x f(x)$

$\mu_1' = \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!}$

$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{x \lambda^x}{x!}$

$= e^{-\lambda} \sum_{x=1}^{\infty} \frac{x (\lambda^{x-1}) \cdot \lambda}{x(x-1)!}$

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$$= e^{-\lambda} \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda} \left[1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots \right]$$

$$= \lambda . e^{-\lambda} e^{\lambda}$$

$$E(X) = \lambda$$

$$\therefore \quad \mu_1' = \lambda$$

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$$\mu_2' = E(X^2) = \sum x^2 f(x)$$

$$= \sum_{x=0}^{\infty} [x(x-1) + x] . \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{\lambda^x e^{-\lambda}}{x!} + \sum_{x=0}^{\infty} x . \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= e^{-\lambda} \sum_{x=2}^{\infty} x(x-1) \frac{\lambda^{x-2} . \lambda^2}{x(x-1)(x-2)!} + \lambda$$

$$= e^{-\lambda} \lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda$$

$$= \lambda^2 e^{-\lambda} \left[1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots \right] + \lambda$$

$= \lambda^2 e^{-\lambda} e^{\lambda} + \lambda$

$= \lambda^2 + \lambda$

$\mu_2' = \lambda + \lambda^2$

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$\mu_3' = \sum x^3 f(x)$

$= \sum \left(x(x-1)(x-2) + 3x(x-1) + x \right) f(x)$

$= \sum x(x-1)(x-2) f(x) + 3 \sum x(x-1) f(x) + \sum x f(x)$

$\mu_3' = \lambda^3 e^{-\lambda} \sum_{x=3}^{\infty} x(x-1)(x-2) \cdot \frac{\lambda^{x-3}}{x(x-1)(x-2)(x-3)!} + 3\lambda^2 + \lambda$

$\mu_3' = \lambda^3 e^{-\lambda} \sum_{x=3}^{\infty} \frac{\lambda^{x-3}}{(x-3)!} + 3\lambda^2 + \lambda$

$\mu_3' = \lambda^3 e^{-\lambda} \left\{ 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots \right\} + 3\lambda^2 + \lambda$

$\mu_3' = \lambda^3 e^{-\lambda} e^{\lambda} + 3\lambda^2 + \lambda$

$\mu_3' = \lambda^3 + 3\lambda^2 + \lambda$

$$\mu_4' = \sum x^4 f(x)$$

$$= \sum \left(x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x \right] f(x)$$

$$= \sum x(x-1)(x-2)(x-3) f(x) + 6 \sum x(x-1)(x-2) f(x)$$

$$+ 7 \sum x(x-1) f(x) + \sum x f(x)$$

$$= \lambda^4 e^{-\lambda} \sum_{x=4}^{\infty} x(x-1)(x-2)(x-3) \frac{\lambda^{x-4}}{x(x-1)(x-2)(x-3)(x-4)!}$$

$$+ 6\lambda^3 + 7\lambda^2 + \lambda$$

$$= \lambda^4 e^{-\lambda} \sum_{x=4}^{\infty} \frac{\lambda^{x-4}}{(x-4)!} + 6\lambda^3 + 7\lambda^2 + \lambda$$

$$= \lambda^4 e^{-\lambda} \left\{ 1 + \lambda + \frac{\lambda^2}{2!} + \cdots \right\} + 6\lambda^3 + 7\lambda^2 + \lambda$$

$$= \lambda^4 e^{-\lambda} e^{\lambda} + 6\lambda^3 + 7\lambda^2 + \lambda$$

$\therefore$

$$\mu_4' = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$$

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Moments about Mean:-

$\mu_1 = 0$

$\mu_2 = \mu_2' - (\mu_1')^2$

$= d^2 + d - d^2$

$= d$

$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3$

$= d^3 + 3d^2 + d - 3(d)(d + d^2) + 2d^3$

$= d^3 + 3d^2 + d - 3d^2 - 3d^3 + 2d^3$

$\mu_3 = d$

$\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$\mu_4 = d^4 + 6d^3 + 7d^2 + d - 4(d)(d^3 + 3d^2 + d)$
$+ 6(d + d^2)(d)^2 - 3d^4$

$\mu_4 = d^4 + 6d^3 + 7d^2 + d - 4d^2 - 12d^3 - 4d^4$
$+ 6d^3 + 6d^4 - 3d^4$

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$\mu_4 = 3d^2 + d$

Now

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}$$

$$= \frac{m^2}{m^3}$$

$$\beta_1 = \frac{1}{m}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} \Rightarrow \frac{3m^2+m}{m^2}$$

$$\beta_2 = \frac{3m^2}{m^2} + \frac{m}{m^2}$$

$$\beta_2 = \left(3 + \frac{1}{m}\right)$$

As Required.

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Problem:- Find Moment Generating function of Poisson Distribution about Mean.

Solution:- As we know that

$$M_{\bar{x}}(t) = E\left[e^{t(x-\bar{x})}\right]$$

In case of Poisson Dist

$$\bar{x} = m$$

$$M_{\bar{x}}(t) = E\left[e^{t(x-m)}\right]$$

$$= E\left[e^{tx-tm}\right]$$

$$= e^{-tm} E(e^{tx})$$

$$= e^{-tm} \{\Sigma e^{tx} f(x)\}$$

$$= e^{-tm} \{\Sigma e^{tx} f(x)\}$$

$$= e^{-tm} \cdot e^{m(e^t-1)}$$

$$= e^{-mt-m+me^t}$$

$$= e^{-mt-m+m(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots)}$$

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$\therefore$

$$M_{\bar{x}}(t) = e^{m\left(\frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\cdots\right)}$$

As Required.

Wish You GOOD LUCK!

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5- Geometeric Distribution

Geometeric Experiment:-

When an Experiment consists of independent trials with probability "p" of success & trials are repeated until the first success occures is called Geometeric Experiment. In other words a Geometeric Experiment has the following four properties.

1) The outcome of each Trial may be classified into one of two categories, success & failure.

2) The probability p of success remains constant for all trials.

3) The successive trials are independent.

4) The Experiment is repeated a variable number of times until the first success is obtained.

If X represents the number of Trials needed for the first success, then X is called a geometeric r.v

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and its Probability Distribution is called Geometeric Distribution. The Geometeric Distribution derives its name from the fact that its successive terms constitute a geometeric Progression.
Since Geometeric r.v represents how long one has to wait for a success, it is also called a waiting time random variable.

Problem:- Derive Geometeric Distribution.

Solution:-
Let r.v X denotes the number of trials required upto & including the first success of an event. The X takes the values 1,2,3,-------, ∞
Since X = x iff the first (x−1) trials results in failure & x^{th} trial yields a success we therefor have the probability Dist,

$$P[X=x] = f(x) = g(x;p) = q^{x-1}p$$

$$x = 1, 2, -------, \infty$$

where p is parameter.

As Required.

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Problem:- Show that Mean & Var of Geometeric Distribution are

$$\mu = 1/p \quad \& \quad \sigma^2 = q/p^2$$

OR

Find Mean & Var of G.D.

Solution:-

$$\text{Mean} = E(X) = \sum x f(x)$$

$$= \sum_{x=1}^{\infty} x\, q^{x-1} p$$

$$= p + 2qp + 3q^2 p + 4q^3 p + \cdots\cdots$$

$$= p(1 + 2q + 3q^2 + 4q^3 + \cdots\cdots)$$

$$= p(1-q)^{-2}$$

Binomial series: $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots$

$$= p(p^{-2}) \Rightarrow \frac{p}{p^2}$$

$$\mu = \frac{1}{p}$$

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$E(X^2) = \sum x^2 f(x)$

$= \sum_{x=1}^{\infty} x^2 q^{x-1} p$

$= (p + 4qp + 9q^2p + 16q^3p + \cdots\cdots)$

$= p(1 + 4q + 9q^2 + 16q^3 + \cdots\cdots)$

$= p\Big[(1 + 3q + 6q^2 + 10q^3 + \cdots\cdots)$

$+ (q + 3q^2 + 6q^3 + \cdots\cdots)\Big]$

$= p\left[(1-q)^{-3} + q(1 + 3q + 6q^2 + \cdots\cdots)\right]$

$= p\left[(1-q)^{-3} + q(1-q)^{-3}\right]$

$= p\left[p^{-3} + q\,p^{-3}\right]$

$= p^{-2}\left[1 + q\right]$

$= \frac{1}{p^2} + \frac{q}{p^2}$

Now,

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$$Var(x) = E(x^2) - (E(x))^2$$

$$= \frac{1}{p^2} + \frac{q}{p^2} - \frac{1}{p^2}$$

$$\sigma^2 = \frac{q}{p^2}$$

As Required.

Problem: Find M.G.F of Geometric Distribution & then using m.g.f find Mean & Var.

Solution:- By def

$$M_x(t) = E(e^{tx})$$

$$= \sum e^{tx} f(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} q^{x-1} p$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} (qe^t)^x$$

$$= \frac{p}{q}\left(qe^t + (qe^t)^2 + (qe^t)^3 + \cdots\cdots\right)$$

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$$= \frac{p}{q} \cdot qe^t \left(1 + qe^t + (qe^t)^2 + \cdots \sim\right)$$

$$= pe^t \left(1 - qe^t\right)^{-1}$$

$$= \frac{pe^t}{1 - qe^t}$$

Hence

$$M.g.f = \frac{pe^t}{1 - qe^t}$$

which is Required.

Now,

$$\mu_1' = E(X) = \frac{d}{dt} M_X(t) \Big|_{t=0}$$

$$\frac{d}{dt} M_X(t) = \frac{d}{dt} \left(pe^t (1 - qe^t)^{-1} \right)$$

$$= pe^t (1 - qe^t)^{-1} + pe^t (1 - qe^t)^{-2} qe^t$$

$$\mu_1' = \frac{d}{dt} (M_X(t)) \Big|_{t=0} = p(1-q)^{-1} + pq(1-q)^{-2}$$

$$= p(p)^{-1} + pq(p^{-2})$$

$$= 1 + qp^{-1}$$

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$$= 1 + \frac{q}{p}$$

$$= \frac{p+q}{p}$$

$$\mu_1' = \frac{1}{p}$$

Now,

$$\mu_2' = \frac{d^2}{dt^2} M_x(t) \Big|_{t=0}$$

$$\frac{d}{dt} M_x(t) = pe^t (1-qe^t)^{-1} + pqe^{2t}(1-qe^t)^{-2}$$

$$\frac{d^2}{dt^2} M_x(t) = pe^t (1-qe^t)^{-1} + qpe^{2t}(1-qe^t)^{-2}$$
$$+ 2qpe^{2t}(1-qe^t)^{-2} + qpe^{2t}(-2)(1-qe^t)^{-3}(-qe^t)$$

$$\frac{d^2}{dt^2} M_x(t) \Big|_{t=0} = p(1-q)^{-1} + qp(1-q)^{-2} + 2pq(1-q)^{-2}$$
$$+ qp(2q)(1-q)^{-3}$$

$$= pp^{-1} + qpp^{-2} + 2pq\,p^{-2} + 2pq^2p^{-3}$$

$$= 1 + qp^{-1} + 2qp^{-1} + 2q^2p^{-2}$$

$$= 1 + \frac{q}{p} + 2\frac{q}{p} + 2\frac{q^2}{p^2}$$

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$$= 1 + \frac{2q}{p} + \frac{q}{p} + \frac{2q^2}{p^2}$$

$$= 1 + \frac{q}{p}\left[3 + \frac{2q}{p}\right]$$

$$= 1 + \frac{q}{p}\left[\frac{3p+2q}{p}\right]$$

$$= 1 + \frac{q}{p}\left[\frac{p+2p+2q}{p}\right]$$

$$= 1 + \frac{q}{p}\left[\frac{p+2}{p}\right]$$

$$= \frac{p^2 + qp + 2q}{p^2}$$

$$= \frac{p(p+q) + 2q}{p^2}$$

$$= \frac{p+2q}{p^2}$$

$$= \frac{1}{p} + \frac{2q}{p^2} = \frac{p+q+q}{p^2}$$

$$\mu_2' = \frac{1+q}{p^2} \Rightarrow \frac{1}{p^2} + \frac{q}{p^2}$$

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Now,

$$Var(X) = \mu_2' - (\mu_1')^2$$

$$= \frac{1}{p^2} + \frac{q}{p^2} - \frac{1}{p^2}$$

$$= \frac{q}{p^2} \quad \text{As Required.}$$

Problem:- Find cumulant Generating Function of Geometric Distribution & find 1st 4 cumulants.

Solution:- As we know that

$$M_X(t) = \frac{pe^t}{1-qe^t}$$

$$= pe^t (1-qe^t)^{-1}$$

$$= pe^t \left[1-(1-p)e^t\right]^{-1}$$

$$= pe^t \left[1 - e^t + pe^t\right]^{-1}$$

$$= pe^t (pe^t)^{-1} \left[\frac{1}{pe^t} + 1 - \frac{1}{p}\right]^{-1}$$

$$= \left[1 - \frac{1}{p} + \frac{e^{-t}}{p}\right]^{-1}$$

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$M_X(t) = \left[1 - \frac{1}{p}(1-e^{-t})\right]^{-1}$

$K(t) = \log_e M_X(t) = \log\left[1 - \frac{1}{p}(1-e^{-t})\right]^{-1}$

$= -\log_e\left[1 - \frac{1}{p}(1-e^{-t})\right]$

$= -\log_e\left[1 - \frac{1}{p}\left(t - \frac{t^2}{2!} + \frac{t^3}{3!} - \frac{t^4}{4!} + \cdots\right)\right]$

As we know

$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$

$\frac{1}{p}\left(t - \frac{t^2}{2!} + \frac{t^3}{3!} - \frac{t^4}{4!} + \cdots\right) + \frac{1}{2p^2}\left(t - \frac{t^2}{2!} + \frac{t^3}{3!} - \cdots\right)^2$

$+ \frac{1}{3p^3}\left(t - \frac{t^2}{2!} + \cdots\right)^3 + \frac{1}{4p^4}\left(t + \cdots\right)^4$

As K_r = Co-efficients of $\frac{t^r}{r!}$ in the Expansion of $\log_e[M_X(t)]$.

$\therefore \quad K_1 = 1/p, \quad K_2 = q/p^2$

$K_3 = \frac{1}{p} - \frac{3}{p^2} + \frac{2}{p^3} = \frac{2 - 3p + p^2}{p^3}$

$\Rightarrow \quad K_3 = \frac{(1-p)(2-p)}{p^3} = \frac{q(1+q)}{p^3}$

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$$K_4 = \frac{-1}{p} + \frac{4}{p^2} + \frac{3}{p^2} - \frac{12}{p^3} + \frac{6}{p^4}$$

$$= \frac{-p^3 + 6 - 12p + 7p^2}{p^4}$$

$$= \frac{(1-p)(6-6p+p^2)}{p^4}$$

$$= \frac{q}{p^4}\left[(1-p)^2 + 1 + 4(1-p)\right]$$

$$= \frac{q}{p^4}\left[q^2 + 1 + 4q\right]$$

$$K_4 = \frac{q}{p^4}\left[4q + q^2 + 1\right]$$ As Required

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Problem:- Show that $P[X = x] = q^{x-1}p$ is a Probability Dist.

Solution:- It is obviously,

$$P(X = x) \geq 0$$

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Now

$$\sum P[X=x] = \sum_{x=1}^{\infty} q^{x-1} p$$

$$= p + qp + q^2p + q^3p + q^4p + \cdots$$

$$= p[1 + q + q^2 + q^3 + q^4 + \cdots]$$

$$= p(1-q)^{-1}$$

$$= p(p)^{-1}$$

$$= 1$$

Hence proved $P[X=x] = q^{x-1}p$ is a Probability Distribution.

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6. Negative Binomial Distribution

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In the Binomial Experiments, the number of success varies & number of trials fixed. But there are experiments in which number of trials varies to produce the fixed number success. Such Experiments are called Negative Binomial Experiment. In other words a negative Binomial Experiment possesses the following four Properties.

i) The outcomes of each trial may be classified into one of two categories: success (S) & failure (F).

ii) The Probability of success denoted by P, remains constant for all trials.

iii) The Successive trials are all independent.

iv) The Experiment is Repeated a variable number of times to obtain a fixed number of success.

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When X denote number of trials to produce k successes in a negative Binomial Experiment, it is called negative Binomial variable and its p.d is called Negative Binomial Distribution.

When the negative Binomial r.v X assumes a value "x" on which kth success occurs, the negative Binomial Dist is given by,

$$P[X=x] = \binom{x-1}{k-1} p^k q^{x-k} \qquad x = k, k+1, k+2, \ldots$$

The negative Binomial p.d has two Parameters k & p > 0 and is generally denoted by,

$$b^*(x; k, p)$$

The Negative Binomial Dist is positively skewed.

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Theorem:- In Negative Binomial Dist., the Probability functions sums to one.

Proof:-

$$\sum b^*(x; k, p) = \sum_{x=k}^{\infty} \binom{x-1}{k-1} p^k q^{x-k}$$

$x = k, k+1, \ldots$

let $y = x - k \Rightarrow x = y + k$

when $x = k \Rightarrow y = 0$

when $x = \infty \Rightarrow y = \infty$ then

$$\text{Sum of Probabilities} = \sum_{y=0}^{\infty} \binom{y+k-1}{k-1} p^k q^y$$

$, y = 0, 1, 2, \ldots$

$$= p^k \sum_{y=0}^{\infty} \binom{y+k-1}{k-1} q^y$$

$$= p^k \left[\binom{k-1}{k-1} q^0 + \binom{k}{k-1} q + \binom{k+1}{k-1} q^2 + \cdots \right]$$

$$= p^k \left[q^0 + k\, q + \frac{k(k+1)}{2!} q^2 + \cdots \right]$$

$$= p^k (1-q)^{-k}$$

$$= p^k p^{-k}$$

$$= 1$$ As Required.

Hint *

$$\frac{(k+1)!}{(k-1)!\,(2!)} = \frac{(k+1)(k)(k-1)!}{(k-1)!\,2!} = \frac{k(k+1)}{2!}$$ *

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Remark:- The distribution takes its name from the fact that $\binom{x-1}{k-1} p^k q^{x-k}$ is a term in the Expansion of $p^k(1-q)^{-k}$, a binomial with negative index. Thus the probabilities at $k+h$, $(k+1)+h$, $(k+2)+h$ ------ trials are

$$p^k\left[1, qk, \frac{k(k+1)}{2} q^2, ------\right]$$

This distribution is sometimes also called the Pascal Distribution. The distribution is found to occure in many biological situations & inverse sampling from a Binomial population.

Derivation of Negative Binomial Distribution:-

The negative Binomial Distribution can be derived in various ways. The following derivation is based on the Bernulli's Trials.
To find an Expression for the probability that x trials are made to acheive k Successes

with condition that the last trial must be success we proceed as follows,

A sequence containing k success in exactly x independent trials with the condition that it ends in a success can be obtained as,

$$\underbrace{S \cdot S \cdots\cdots S}_{(k-1)\text{ times}} \quad \underbrace{F \cdot F \cdot F \cdots\cdots F}_{(x-k)\text{ times}} \cdot S$$

Thus probability of success on the xth trial preceded by $(k-1)$ successes & $(x-k)$ failures is,

$$p^{k-1} q^{x-k}\, p = p^k q^{x-k}$$

Since the last trial must be success therefore the total number of mutually exclusive ways in which $(k-1)$ successes & $(x-k)$ failures preceding the last success can occur in any order is given by

$$\binom{x-1}{k-1}$$

Hence the formula for the probability that kth success occurs on x^{th} trial is,

$$P(X=x) = \binom{x-1}{k-1} p^k q^{x-k} \quad ; \; x = k, k+1, \ldots$$

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Example 1) A Person throws a Pair of fair dice. what is Probability that he will get total of "7" for the second time on the eights throw?

Solution: The Probability of getting total of 7 is 6/36 i.e,

$$p = 1/6$$

$$q = 1 - \frac{1}{6} = 5/6$$

Since number of success is fixed therefor the negative Binomial Distribution with $k = 2$ (second success) & $x = 8$ used

Hence,

$$P(X = x) = \binom{x-1}{k-1} p^k q^{x-k}$$

$$P(X = 8) = \binom{8-1}{2-1} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{8-2}$$

$$= \binom{7}{1} \left(\frac{1}{36}\right) \left(\frac{5}{6}\right)^6$$

$$= \frac{7}{36} \cdot \frac{5^6}{36}$$

$$= 0.0651$$

As Required.

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Example 2) Three People each toss a coin and odd man Pays for the coffee. If the coins all shows heads or all shows tails, they are tossed again. What is Probability that a decision is reached in five tosses or fewer.

Solution:- To reach a decision on any trial, the coins must result in either 2 heads & 1 tail or 2 tails and 1 head. The Probability of these events is computed by the binomial Distribution.

$$P(2\text{ heads}, 1\text{ tail}) = \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) = \frac{3}{8}$$

$$P(1\text{ head}, 2\text{ tails}) = \binom{3}{1}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 = 3/8$$

Hence Probability of reaching a decision is

$$\frac{3}{8} + \frac{3}{8} = \frac{6}{8} = 0.75$$

Next we find the Probability that a decision is reached in 5 tosses or fewer

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We therefor compute the required probability by means of the negative binomial distribution where

$$K = 1, \quad P = 0.75, \quad q = 0.25$$

& $x \leq 5$ (the number of trials)

$$P(X \leq 5) = \sum_{x=1}^{5} \binom{x-1}{k-1} (0.75)^k (0.25)^{x-k}$$

$$= 0.75 + 0.1875 + 0.0469 + 0.0117 + 0.0029$$

$$= 0.9990$$

As Required

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Example 3) The Probability that a swimmer will succeed in swimming across a lake is 0.4. What is probability that 10th swimmer is fourth one to cross the lake.

Solution:- i) There are two possible outcomes i.e, the swimmer will or will not cross the lake.

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ii) Probability of success in swimming across the lake for each swimmer is $P = 0.4$

iii) The swimmer will cross the lake independent.

iv) The Experiment is repeated 10 times to get 4th success.

$\therefore$ Negative Binomial Distribution with $k = 4$, $x = 10$ is appropriate

$p = 0.4$

$q = 0.6$

$k = 4$

$x = 10$

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$$b^*(x; k, p) = \binom{x-1}{k-1} p^k q^{x-k}$$

$$= \binom{10-1}{4-1} (0.4)^4 (0.6)^{10-4}$$

$$= \binom{9}{3} (0.4)^4 (0.6)^6$$

$$= 0.10033$$ As Required.

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Example 4) Find the Probability that a person flipping a coin gets the third head on the seventh flip.

Solution:-

$$p = 1/2$$

$$q = 1/2$$

$$n = 7 \text{ (Seventh trial)}$$

$$k = 3 \text{ (third success)}$$

$$b^*(n; k, p) = \binom{n-1}{k-1} p^k q^{n-k}$$

$$= \binom{7-1}{3-1} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{7-3}$$

$$= \binom{6}{2} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^4$$

$$= 0.1172$$

As Required.

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Example 5) The Probability that a person will install a black telephone in a residence is estimated to be 0.3. Find the Probability that the 10th phone installed in a new Sub-Divison is the 5th Black Phone.

Solution)
$p = 0.3$
$q = 0.7$
$k = 5$ (5th success)
$x = 10$ (10th telephone)

$$b^*(x; k, p) = \binom{x-1}{k-1} p^k q^{x-k}$$

$$= \binom{10-1}{5-1} (0.3)^5 (0.7)^{10-5}$$

$$= \binom{9}{4} (0.3)^5 (0.7)^5$$

$$= 0.0515$$

As Required.

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Example 6) Find the Probability that a person tossing three coins will get either all heads or all tails for the second time on the 5th Toss.

Solution:- If we throw 3 coins at a time, the sample space is,

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

$n(S) = 2^3 = 8$

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Let A denotes the event that all heads or all trails appear.

$$A = \{HHH, TTT\}$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{2}{8} = \frac{1}{4}$$

$$p = \frac{1}{4}$$

$$q = 3/4$$

$$k = 2 \text{ (second success)}$$

$$x = 5$$

$$b^*(x; k, p) = \binom{x-1}{k-1} p^k q^{x-k}$$

$$= \binom{5-1}{2-1} \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^{5-2}$$

$$= \binom{4}{1} \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^3$$

$$= 4 \cdot \frac{1}{6} \cdot \frac{27}{64}$$

$$= \frac{27}{256}$$ As Required.

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*Example 7) If the Probability that a person will believe a rumour about the retirement of a certain Politician is 0.25. What is Probability that

i) the 6th person to hear the rumour will be the first to beleive it.

ii) The 12th person to hear the rumour will the 4th to beleive it.

Solution:- i) $p = 0.25$

$q = 0.75$

$x = 6$

$k = 1$

$$b^*(x; k, p) = \binom{x-1}{k-1} p^k q^{x-k}$$

$$= \binom{6-1}{1-1} (0.25)^1 (0.75)^5$$

$$= 0.059$$

ii) $p = 0.25$

$q = 0.75$

$x = 12$

$k = 4$

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$$b^*(x;k,p) = \binom{x-1}{k-1} p^x q^{x-k}$$

$$= \binom{12-1}{4-1} (0.25)^4 (0.75)^{12-4}$$

$$= \binom{11}{3} (0.25)^4 (0.75)^8$$

$$= 0.0645$$

As Required.

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Problem) Derive Mean & Variance of Negative Binomial Distribution and show that its Mean is Less than its Variance.

Solution:- We find Mean & Variance by deriving the m.g.f of the Negative Binomial Distribution.

The m.g.f about the origin is

$$M_o(t) = E(e^{tx})$$

$$= \sum e^{tx} f(x)$$

$$= \sum_{x=k}^{\infty} e^{tx} \binom{x-1}{k-1} p^k q^{x-k}$$

$$= \sum_{x=0}^{\infty} e^{tx} \binom{x+k-1}{k-1} p^k q^x$$

$$= p^k \sum_{x=0}^{\infty} \binom{x+k-1}{k-1} (qe^t)^x$$

$$= p^k \left[\binom{k-1}{k-1} (qe^t)^0 + \binom{k}{k-1} (qe^t) + \binom{k+1}{k-1} (qe^t)^2 + \cdots \right]$$

$$= p^k \left[1 + kqe^t + \frac{k(k+1)}{2!} (qe^t)^2 + \cdots \right]$$

$$= p^k (1 - qe^t)^{-k}$$

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$$\Rightarrow \quad M_0(t) = p^k (1 - qe^t)^{-k}$$

Now,

$$\text{Mean} = E(X) = \left[\frac{dM_0(t)}{dt} \right]_{t=0}$$

$$= p^k \left[-k (1 - qe^t)^{-k-1} (-qe^t) \right]_{t=0}$$

$$= \left(p^k k qe^t (1 - qe^t)^{-k-1} \right)_{t=0}$$

$$= p^k kq (1-q)^{-k-1}$$

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$= p^{k} k q \; p^{-k-1}$

$= p^{k-k-1} kq$

$= p^{-1} k q$

Mean: $= \dfrac{kq}{p}$

Now,

$E(X^2) = \left(\dfrac{d^2 M_o(t)}{dt^2}\right)_{t=0}$

$= \left(p^{k} k q e^{t} (1-qe^{t})^{-k-1} + p^{k} k q e^{t} (-k-1)(1-qe^{t})^{-k-2}(-qe^{t})\right)_{t=0}$

$= \left[p^{k} k q e^{t} (1-qe^{t})^{-k-1} + p^{k} k q e^{t} \cdot -(k+1)(1-qe^{t})^{-k-2}(-qe^{t})\right]_{t=0}$

$= \left[p^{k} k q e^{t} (1-qe^{t})^{-k-1} + p^{k} k(k+1) \cdot q^{2} e^{2t} (1-qe^{t})^{-k-2}\right]_{t=0}$

$= p^{k} k q (1-q)^{-k-1} + p^{k} k(k+1) q^{2} (1-q)^{-k-2}$

$= p^{k} k q \, p^{-k-1} + p^{k} k(k+1) q^{2} (p)^{-k-2}$

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$$= p^{k-k-1} \, kq + p^{k-k-2} \, k(k+1) q^2$$

$$= \frac{kq}{p} + \frac{k(k+1) q^2}{p^2}$$

$$= \frac{kq}{p} + \frac{k^2 q^2}{p^2} + \frac{kq^2}{p^2}$$

Now

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{kq}{p} + \frac{k^2 q^2}{p^2} + \frac{kq^2}{p^2} - \frac{k^2 q^2}{p^2}$$

$$= \frac{kq}{p} + \frac{kq^2}{p^2}$$

$$= \frac{kq}{p^2} (p + q)$$

$$Var(X) = \frac{kq}{p^2}$$

As Required.

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The Variance will be greater than mean if

$$\frac{Kq}{p^2} > \frac{Kq}{p}$$

$$\frac{1}{p^2} > \frac{1}{p}$$

$$\frac{1}{p} > 1$$

$$1 > p$$

$$\Rightarrow p < 1$$

which is obviously true.

Hence we observe that the Variance of the Negative Binomial distribution is greather its Mean.
This is an importent Result of this distribution.

As Required.

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*Find First Four Cumulants for Negative Binomial Distribution.

As,

$$M_o(t) = p^k (1 - qe^t)^{-k}$$

$$= \left(\frac{p}{1-qe^t}\right)^k$$

$$= \left(\frac{p}{p+q-qe^t}\right)^k$$

$$= \left[\frac{p}{p+(-q(e^t-1))}\right]^k$$

$$= \left[\frac{p}{p-q(e^t-1)}\right]^k$$

$$= \left[\frac{1}{1-\frac{q}{p}(e^t-1)}\right]^k$$

$$= \left[1 - d(e^t-1)\right]^{-k} \quad \text{where } d = q/p$$

Now

$$K(t) = \log_e M_o(t)$$

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$$= \log\left[1 - d(e^t - 1)\right]^{-k}$$

$$= -k \log\left(1 - d(e^t - 1)\right)$$

$$= k\left[d(e^t - 1) + \frac{d^2(e^t-1)^2}{2} + \frac{d^3(e^t-1)^3}{3} + \frac{d^4(e^t-1)^4}{4} + \cdots\right]$$

$$= k\left[d\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots - 1\right) + \frac{d^2}{2}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots - 1\right)^2 + \frac{d^3}{3}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots - 1\right)^3 + \frac{d^4}{4}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots - 1\right)^4 + \cdots\right]$$

$$= k\left[d\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right) + \frac{d^2t^2}{2}\left(1 + \frac{t}{2!} + \frac{t^2}{3!} + \frac{t^3}{4!} + \cdots\right)^2 + \frac{1}{3}d^3t^3\left(1 + \frac{t}{2!} + \cdots\right)^3 + \frac{1}{4}d^4t^4 + \cdots\right]$$

$$= k\left[dt + (d+d^2)\frac{t^2}{2!} + (d+3d^2+2d^3)\frac{t^3}{3!} + (d+7d^2+12d^3+6d^4)\frac{t^4}{4!} + \cdots \right]$$

$$= k\left[\frac{q}{p}t + \frac{q}{p^2}\frac{t^2}{2!} + \frac{q(1+q)}{p^3}\frac{t^3}{3!} + \frac{q(1+4q+q^2)}{p^4}\frac{t^4}{4!} + \cdots \right]$$

Identify the co-efficient we get

$$K_1 = kq/p$$

$$K_2 = kq/p^2$$

$$K_3 = \frac{kq(1+q)}{p^3}$$

$$K_4 = \frac{kq(1+4q+q^2)}{p^4}$$

As Required

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7. HyperGeometeric Probability Distribution

There are many Experiments in which the Condition of independence is violated and the Probability of success does not remains constant for all trials. Such Experiments are called HyperGeometeric Experiments. In other words a HyperGeometeric Experiment has the following Properties.

i) The outcome of each trial may be classified into one of two catagories, Success & Failure.

ii) The Probability of success changes on each trial.

iii) The successive trials are dependent.

iv) The Experiment is repeated a fixed number of times.

The number of successes X in a Hypergeometeric Experiment is called a Hypergeometeric r.v & its distribution is called Hyper Geometeric Distribution.

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When a hypergeometeric r.v X assumes a value 'x' the hypergeometeric p.d is given by the formula,

$$P(X=x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} \quad x = 0,1,2,\text{——},n \;\&\; x = 0,1,2,\text{----},K$$

where

N = number of units in the set or population (a +ve integer)

n = number of units in the subset or sample (a +ve integer)

K = no of successes in the set or population (a non-negative integer less than or equal to N).

The HyperGeometeric Probability Distribution has three parameters N, n & K & in generally denoted by h(x; N, n, K). The H.G p.d is approporiate when

i) A random sample of size 'n' is drawn without replacement from a finite population of N units.

ii) K of the units are of one kind (classified OR success) and the remaining N-K of another kind (classified as failure).

*Derivation of HyperGeometeric Distribution:-

Let a Population Consists of N items out of which "K" items are of interest to us i.e, there are K successes & N-K items are of not of interest to us i.e, there are "N-K" failures.

Let a sample of size "n" be selected from N without replacement, this can be done in $\binom{N}{n}$ ways.

Let the number of successes in this sample be $(X=x)$ then the number of failures in the sample is $n-x$.

The x number of successes can be selected from K successes in the population in $\binom{K}{x}$ ways.

Similarly $\binom{N-K}{n-x}$ for failures. So number of sample points in the Event of x successes & $n-x$ failures in the sample is, $\binom{K}{x}\binom{N-K}{n-x}$

∴ The probability of x- successes & $n-x$ failure is,

$$P(X=x) = f(x) = h(x; N, n, K) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

Assuming $x \leq K$

As Required.

* Problem:- Prove that the sum of HyperGeometric Probabilities is one.

Solution:- The HyperGeometric p.d is given by,

$$P(X=x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} \qquad \begin{matrix} 0 \leq x \leq n \\ 0 \leq x \leq K \end{matrix}$$

$$\sum_{x=0}^{n} P[X=x] = \frac{1}{\binom{N}{n}} \sum_{x=0}^{n} \binom{K}{x}\binom{N-K}{n-x} \rightarrow (1)$$

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As $\sum_{j=0}^{m} \binom{a}{j}\binom{b}{m-j} = \binom{a+b}{m}$

So ① $\Longrightarrow$

$$\sum_{x=0}^{n} P[X=x] = \frac{1}{\binom{N}{n}} \binom{N-K+K}{n-x+x}$$

$$= \frac{\binom{N}{n}}{\binom{N}{n}}$$

$$= 1$$

So, Sum of Hyper Geometeric Probabilities is one.

Mean & Variance of Hyper Geometeric Distribution:-

For Hyper Geometeric Distribution

$$p(x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} \quad ; x = 0, 1, 2, \ldots, n$$

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Mean = $E(x) = \sum x P(x)$

$$= \sum_{x=0}^{n} x \cdot \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

$$= \frac{1}{\binom{N}{n}} \sum_{x=0}^{n} x \binom{K}{x}\binom{N-K}{n-x}$$

$$= \frac{1}{\binom{N}{n}} \sum_{x=0}^{n} x \cdot \frac{K(K-1)!}{x(x-1)!\,(K-x)!} \binom{N-K}{n-x}$$

$$= \frac{K}{\binom{N}{n}} \sum_{x=1}^{n} \frac{(K-1)!}{(x-1)!\,(K-x)!} \binom{N-K}{n-x}$$

$$= \frac{K}{\binom{N}{n}} \sum_{x-1=0}^{n} \binom{K-1}{x-1}\binom{N-K}{n-x}$$

Put $x-1 = y \Rightarrow x = 1+y$

$x = 1 \Rightarrow y = 0$

$x = n \Rightarrow y = n-1$

$$= \frac{K}{\binom{N}{n}} \sum_{y=0}^{n-1} \binom{K-1}{y}\binom{N-K}{n-1-y}$$

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$$= \frac{K}{\binom{N}{n}} \binom{N-1}{n-1} \quad \because \sum_{j=0}^{m} \binom{a}{j}\binom{b}{m-j} = \binom{a+b}{m}$$

$$= \frac{K}{\frac{N!}{n!\,(N-n)!}} \cdot \frac{(N-1)!}{(n-1)!\,(N-n)!}$$

$$= \frac{K}{N!}\, n!\,\cancel{(N-n)!}\; \frac{(N-1)!}{(n-1)!\,\cancel{(N-n)!}}$$

$$= \frac{K}{N(N-1)!} \cdot n(n-1)! \cdot \frac{(N-1)!}{(n-1)!}$$

$$= \frac{Kn}{N}$$

Mean $= n \cdot K/N$

Mean $= np$. $\quad \therefore \frac{K}{N} = p$

AS Required

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Now,

$$E(X^2) = E\left[X(X-1) + X\right]$$

$$= E\left[X(X-1)\right] + E(X)$$

$$= \sum_{x=0}^{n} x(x-1)\, P(x) + n\frac{K}{N} \longrightarrow (1)$$

Now Consider,

$$= \sum_{x=0}^{n} x(x-1) \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

$$= \frac{1}{\binom{N}{n}} \sum_{x=0}^{n} x(x-1) \frac{K!}{x!(K-x)!} \binom{N-K}{n-x}$$

$$= \frac{1}{\binom{N}{n}} \sum_{x=2}^{n} \frac{K(K-1)(K-2)!}{(x-2)!\,(K-x)!} \binom{N-K}{n-x}$$

$$= \frac{K(K-1)}{\binom{N}{n}} \sum_{x-2=0}^{n} \binom{K-2}{x-2} \binom{N-K}{n-x}$$

Put $x-2 = y \Rightarrow x = y+2$

$x = 2 \Rightarrow y = 0$

$x = n \Rightarrow y = n-2$

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$$= \frac{K(K-1)}{\binom{N}{n}} \sum_{y=0}^{n-2} \binom{K-2}{y}\binom{N-K}{n-2-y}$$

$$= \frac{K(K-1)}{\binom{N}{n}} \binom{N-2}{n-2}$$

$$= \frac{K(K-1)}{\frac{N!}{n!\,(N-n)!}} \frac{(N-2)!}{(n-2)!\,(N-n)!}$$

$$= \frac{nK(n-1)(K-1)}{N(N-1)}$$

So $1 \Rightarrow$

$$E(X^2) = \frac{nK(n-1)(K-1)}{N(N-1)} + \frac{nK}{N}$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{nK(n-1)(K-1)}{N(N-1)} + \frac{nK}{N} - \left(\frac{nK}{N}\right)^2$$

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$$= \frac{nK}{N}\left[\frac{nK-n-K+1}{N-1}+1-\frac{nK}{N}\right]$$

$$= \frac{nK}{N}\left[\frac{NnK-Nn-NK+N+N^2-N-nNK+nK}{N(N-1)}\right]$$

$$= \frac{nK}{N}\left[\frac{N^2-nN-NK+nK}{N(N-1)}\right]$$

$$= \frac{nK}{N}\left[\frac{N(N-n)-K(N-n)}{N(N-1)}\right]$$

$$= \frac{nK}{N}\cdot\frac{N-K}{N}\cdot\frac{N-n}{N-1} \longrightarrow (4)$$

If we put $\frac{K}{N} = p$

$$q = 1-p = 1-K/N$$

$$q = \frac{N-K}{N}$$

So,

$$Var(X) = npq\left(\frac{N-n}{N-1}\right)$$

If we set $\frac{K}{N} = p$, the mean of the HyperGeometeric Distribution

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Coincides with the mean of the Binomial Distribution & variance of HyperGeometeric Distribution is $\left(\frac{N-n}{N-1}\right)$ times the variance of the Binomial Distribution. As Required.

Problem)

Find The Mean & Variance of HyperGeometeric Distribution, Taking

$$P(x) = \frac{\binom{Np}{x}\binom{Nq}{n-x}}{\binom{N}{n}}$$

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Solution:-

$$\text{Mean} = E(X) = \sum x\,P(x)$$

$$= \sum_{x=0}^{n} x \frac{\binom{Np}{x}\binom{Nq}{n-x}}{\binom{N}{n}}$$

$$= \frac{1}{\binom{N}{n}} \sum_{x=0}^{n} x \binom{Np}{x}\binom{Nq}{n-x}$$

$$= \frac{1}{\binom{N}{n}} \sum_{x=0}^{n} x \frac{(Np)!}{x!\,(Np-x)!} \binom{Nq}{n-x}$$

$$= \frac{1}{\binom{N}{n}} \sum_{x=0}^{n} x \frac{Np(Np-1)!}{x(x-1)!(Np-x)!} \binom{Nq}{n-x}$$

$$= \frac{Np}{\binom{N}{n}} \sum_{x-1=0}^{n} \binom{Np-1}{x-1} \binom{Nq}{n-x}$$

Put $x-1 = y \Rightarrow x = 1+y$

$x = 1 \Rightarrow y = 0$

$x = n \Rightarrow y = n-1$

$$= \frac{Np}{\binom{N}{n}} \sum_{y=0}^{n-1} \binom{Np-1}{y} \binom{Nq}{n-1-y}$$

$$= \frac{Np}{\binom{N}{n}} \binom{Np+Nq-1}{n-1}$$

$$= \frac{Np}{\binom{N}{n}} \binom{N-1}{n-1}$$

$$= \frac{Np}{\frac{N!}{n!(N-n)!}} \frac{(N-1)!}{(n-1)!(N-n)!}$$

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$$= \frac{\cancel{N}P}{\cancel{N}(N-1)!} \; n(n-1)!\,(N-n)! \; . \; \frac{(N-1)!}{(n-1)!\,(N-n)!}$$

$$E(X) = np$$

Now,

$$E(X^2) = E\left[X(X-1) + X\right]$$

$$= E(X(X-1)) + E(X)$$

$$= E(X(X-1)) + np \longrightarrow (A)$$

Consider,

$$E\left[X(X-1)\right] = \sum_{x=0}^{n} x(x-1)\,P(x)$$

$$= \sum_{x=0}^{n} x(x-1) \frac{\binom{Np}{x}\binom{Nq}{n-x}}{\binom{N}{n}}$$

$$= \frac{1}{\binom{N}{n}} \sum_{x=0}^{n} x(x-1) \binom{Np}{x}\binom{Nq}{n-x}$$

$$= \frac{1}{\binom{N}{n}} \sum_{x=0}^{n} x(x-1) \frac{Np!}{x(x-1)(x-2)!\,(Np-x)!} \binom{Nq}{n-x}$$

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$$= \frac{1}{\binom{N}{n}} \sum_{x=2}^{n} \frac{NP(NP-1)(NP-2)!}{(x-2)!\,(NP-x)!} \binom{Nq}{n-x}$$

$$= \frac{NP(NP-1)}{\binom{N}{n}} \sum_{x=2}^{n} \binom{NP-2}{x-2} \binom{Nq}{n-x}$$

Now Put $x-2 = y \Rightarrow x = y+2$

$x = 2 \Rightarrow y = 0$

$x = n \Rightarrow y = n-2$

$$= \frac{NP(NP-1)}{\binom{N}{n}} \sum_{y=0}^{n-2} \binom{NP-2}{y} \binom{Nq}{n-y-2}$$

$$= \frac{(NP)(NP-1)}{\binom{N}{n}} \binom{NP+Nq-2}{n-2}$$

$$= \frac{(NP)(NP-1)}{\binom{N}{n}} \binom{N-2}{n-2}$$

$$= \frac{(NP)(NP-1)}{\frac{N!}{n!(N-n)!}} \frac{(N-2)!}{(n-2)!\,(N-n)!}$$

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$$= \frac{NP(NP-1)}{N(N-1)(N-2)!} \; n(n-1)(n-2)!\,(N-n)! \; \frac{(N-2)!}{(n-2)!\,(N-n)!}$$

$$= \frac{NP(NP-1)\,n(n-1)}{N(N-1)}$$

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So $A \Rightarrow$

$$E(x^2) = \frac{NP(NP-1)\,n(n-1)}{N(N-1)} + nP$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{NP(NP-1)\,n(n-1)}{N(N-1)} + nP - n^2P^2$$

$$= nP\left[\frac{(NP-1)(n-1)}{N-1} + 1 - nP\right]$$

$$= nP\left[\frac{NPn - NP - n + 1 + N - 1 - nPN + nP}{N-1}\right]$$

$$= np\left[\frac{N - Np + np - n}{N-1}\right]$$

$$= \frac{np}{N-1}\left[N(1-p) + n(p-1)\right]$$

$$= \frac{np}{N-1}\left[Nq - nq\right]$$

$$= \frac{np}{N-1}\left[(N-n)\,q\right]$$

$$Var(X) = npq\left(\frac{N-n}{N-1}\right)$$

As Required.

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Problem:- Find Mean & Variance of HyperGeometeric Distribution Taking

$$p(x) = \frac{\binom{m}{x}\binom{n}{r-x}}{\binom{m+n}{r}}$$

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Solution:- Mean $= E(x) = \sum x\, P(x)$

$$= \sum_{x=0}^{n} x \cdot \frac{\binom{m}{x}\binom{n}{r-x}}{\binom{m+n}{r}}$$

$$= \frac{1}{\binom{m+n}{r}} \sum_{x=0}^{n} x \cdot \binom{m}{x}\binom{n}{r-x}$$

$$= \frac{1}{\binom{m+n}{r}} \sum_{x=0}^{n} x \frac{m\,(m-1)!}{x(x-1)!\,(m-x)!} \binom{n}{r-x}$$

$$= \frac{m}{\binom{m+n}{r}} \sum_{x=1}^{n} \frac{(m-1)!}{(x-1)!\,(m-x)!} \binom{n}{r-x}$$

$$= \frac{m}{\binom{m+n}{r}} \sum_{x-1=0}^{n} \binom{m-1}{x-1} \binom{n}{r-x}$$

Put $x-1 = y \Rightarrow x = y+1$

$x = 1 \Rightarrow y = 0$

$x = n \Rightarrow y = n-1$

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$$= \frac{m}{\binom{m+n}{r}} \sum_{y=0}^{n-1} \binom{m-1}{y} \binom{n}{r-y-1}$$

$$= \frac{m}{\binom{m+n}{r}} \binom{m+n-1}{r-1}$$

$$= \frac{m}{\frac{(m+n)!}{(m+n-r)!\, r!}} \cdot \frac{(m+n-1)!}{(r-1)!\,(m+n-r)!}$$

$$= \frac{m}{(m+n)\cancel{(m+n-1)!}} \cdot \cancel{(m+n-r)!}\, r\cancel{(r-1)!} \cdot \frac{\cancel{(m+n-1)!}}{\cancel{(r-1)!}\,\cancel{(m+n-r)!}}$$

$$= \frac{rm}{m+n}$$

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$$\Rightarrow E(X) = \frac{rm}{m+n}$$

Now,

$$E(X^2) = E[X(X-1) + X] \Rightarrow E[X(X-1)] + E(X)$$

$$E(X^2) = E(X(X-1)) + \frac{mr}{m+n} \longrightarrow (A)$$

Now Consider,

$$E[X(X-1)] = \sum x(x-1)\, P(x)$$

$$= \sum_{x=0}^{n} x(x-1) \frac{\binom{m}{x}\binom{n}{r-x}}{\binom{m+n}{r}}$$

$$= \frac{1}{\binom{m+n}{r}} \sum_{x=2}^{n} x(x-1) \frac{m(m-1)(m-2)!}{x(x-1)(x-2)!(m-x)!} \binom{n}{r-x}$$

$$= \frac{m(m-1)}{\binom{m+n}{r}} \sum_{x=+2}^{n} \frac{(m-2)!}{(x-2)!(m-x)!} \binom{n}{r-x}$$

$$= \frac{m(m-1)}{\binom{m+n}{r}} \sum_{x-2=0}^{n} \binom{m-2}{x-2}\binom{n}{r-x}$$

Put $x-2 = y \Rightarrow x = y+2$

$x = 2 \Rightarrow y = 0$

$x = n \Rightarrow y = n-2$

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$$= \frac{m(m-1)}{\binom{m+n}{r}} \sum_{y=0}^{n-2} \binom{m-2}{y} \binom{n}{r-y-2}$$

$$= \frac{m(m-1)}{\binom{m+n}{r}} \binom{m+n-2}{r-2}$$

$$= \frac{m(m-1)}{\frac{(m+n)!}{r!(m+n-r)!}} \frac{(m+n-2)!}{(r-2)!(m+n-r)!}$$

$$= \frac{m(m-1)}{(m+n)(m+n-1)(m+n-2)!} \cdot \frac{r(r-1)(r-2)!\,(m+n-2)!}{(r-2)!}$$

$$= \frac{mr(m-1)(r-1)}{(m+n)(m+n-1)}$$

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So (A) $\Rightarrow$

$$E(X^2) = \frac{mr(m-1)(r-1)}{(m+n)(m+n-1)} + \frac{rm}{m+n}$$

Now,

$$Var = E(X^2) - [E(X)]^2$$

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$$Var(X) = \frac{mr(m-1)(r-1)}{(m+n)(m+n-1)} + \frac{rm}{m+n} - \left(\frac{rm}{m+n}\right)^2$$

$$= \frac{rm}{m+n}\left[\frac{(m-1)(r-1)}{(m+n-1)} + 1 - \frac{rm}{m+n}\right]$$

$$= \frac{rm}{m+n}\left[\frac{(m+n)(m-1)(r-1) + (m+n)(m+n-1) - (m+n-1)(rm)}{(m+n-1)(m+n)}\right]$$

$$= \frac{rm}{m+n}\left[\frac{m^2r + m + mnr + n - mr - nr + m^2 + mn - m + mn + n^2 - n - rm^2 - rmn + rm}{(m+n-1)(m+n)}\right]$$

$$= \frac{rm}{m+n}\left[\frac{mn + n^2 - nr}{(m+n)(m+n-1)}\right]$$

$$= \frac{nrm}{m+n}\left[\frac{m-r+n}{(m+n)(m+n-1)}\right]$$

As Required.

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Example 1) An Urn contains 4 red balls & 6 black balls. A sample of 4 balls is selected from the urn without replacement. Let X be the number of red balls contained in the sample, then find Probability distribution for X.

Solution:- $N = 4+6 = 10$

$K = 4$

$n = 4$

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Possible values of x are 0, 1, 2, 3, 4

$P(X=0) = \binom{4}{0}\binom{6}{4} \div \binom{10}{4} = 15/210$

$P(X=1) = \binom{4}{1}\binom{6}{3} \div \binom{10}{4} = 80/210$

$P(X=2) = \binom{4}{2}\binom{6}{2} \div \binom{10}{4} = 90/210$

$P(X=3) = \binom{4}{3}\binom{6}{1} \div \binom{10}{4} = 24/210$

$P(X=4) = \binom{4}{4}\binom{6}{0} \div \binom{10}{4} = 1/210$

Hence H.G. P.d of X is as follows,

x	0	1	2	3	4
$f(x)$	15/210	80/210	90/210	24/210	1/210

As Required.

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Example 2) The names of 5 men & 5 womens are written on slips of paper & placed in a hat. Four names are drawn. what is probability that 2 are men & 2 are women.

Solution:- let x denotes the number of men then

$N = 5+5 = 10$

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$n = 4$

$k = 5$

Here possible values of x are 0, 1, 2, 3, 4

Hence Hyper Geometeric distribution is,

$$P[X=x] = \binom{5}{x}\binom{5}{4-x} \div \binom{10}{4}$$

& Required Probability i.e, $P(X=2)$ is,

$$P[X=2] = \frac{\binom{5}{2}\binom{5}{2}}{\binom{10}{4}}$$

$$= 10/21$$

As Required.

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Example 3) If 5 cards are drawn from a pack of 52 cards what is the probability that 3 will be diamonds.

Solution:- Total no. of card = N = 52

No. of cards drawn = n = 5

let X denotes number of diamonds then

K = 13

$$P(X=x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

$$P[X=x] = \binom{13}{x}\binom{39}{5-x} \div \binom{52}{5}$$

Now,

$$P[X=3] = \binom{13}{3}\binom{39}{2} \div \binom{52}{5}$$

$$= 0.0815$$

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CHAPTER # 6

1- Continous Probability Distribution

1) Uniform OR Rectangular Distribution

The Rectangular OR uniform Distribution is given by p.d.f

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

Problem:-
Find Mean & Variance of Uniform Rectangular Distribution.

Solution:-

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_a^b x \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \left| \frac{x^2}{2} \right|_a^b$$

$$= \frac{1}{2(b-a)} (b^2 - a^2).$$

$$= \frac{1}{2(b-a)} (b+a)(b-a)$$

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$$E(X) = \frac{b+a}{2}$$

Now,

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_a^b x^2 \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \left| \frac{x^3}{3} \right|_a^b$$

$$= \frac{1}{3(b-a)} \{ b^3 - a^3 \}$$

$$= \frac{1}{3(b-a)} (b-a)(b^2+ab+a^2)$$

$$= \frac{a^2+ab+b^2}{3}$$

Now

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{a^2+ab+b^2}{3} - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{a^2+ab+b^2}{3} - \frac{a^2+b^2+2ab}{4}$$

$$= \frac{4a^2+4ab+4b^2-3a^2-3b^2-6ab}{12}$$

$$= \frac{(a-b)^2}{12}$$

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Problem 2) Find First Four moments about origin of the Rectangular Distribution.

Solution:-

$$\mu_r' = \int_{-\infty}^{\infty} x^r f(x)\, dx$$

$$\mu_r' = \int_a^b x^r \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \left| \frac{x^{r+1}}{r+1} \right|_a^b$$

$$\mu_r' = \frac{1}{b-a} \left\{ \frac{b^{r+1} - a^{r+1}}{r+1} \right\}$$

Put $r = 1$

$$\mu_1' = \frac{1}{b-a} \left\{ \frac{b^2 - a^2}{2} \right\}$$

$$\mu_1' = \frac{1}{b-a} \frac{(b+a)(b-a)}{2}$$

$$\mu_1' = \frac{b+a}{2}$$

Put $r = 2$

$$\mu_2' = \frac{1}{b-a} \left[\frac{b^3 - a^3}{3} \right]$$

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$$\mu_2' = \frac{1}{b-a} \cdot \frac{(b-a)(b^2+ab+a^2)}{3}$$

$$\mu_2' = \frac{a^2+ab+b^2}{3}$$

Put $r = 3$

$$\mu_3' = \frac{1}{b-a}\left\{\frac{b^4-a^4}{4}\right\}$$

$$= \frac{1}{b-a} \cdot \frac{(b^2-a^2)(b^2+a^2)}{4}$$

$$= \frac{1}{b-a} \cdot \frac{(b+a)(b-a)(b^2+a^2)}{4}$$

$$\mu_3' = \frac{(b+a)(b^2+a^2)}{4}$$

Put $r = 4$

$$\mu_4' = \frac{1}{b-a}\left(\frac{b^5-a^5}{5}\right)$$

Hence μ_1', μ_2', μ_3' & μ_4' are required first four moments about origin. As Required.

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Problem 3) Show that total Area under the curve of Rectangular Distribution is unity.

Solution:-

$$\int_{-\infty}^{\infty} f(x)dx = \int_a^b f(x)dx$$

$$= \int_a^b \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \int_a^b dx$$

$$= \frac{1}{b-a} \Big|_a^b x$$

$$= \frac{1}{b-a} \cdot (b-a)$$

$$= 1$$

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$$\Rightarrow \int_a^b f(x)\, dx = 1$$

As Required.

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Problem 4) Find Moment Generating Function of Rectangular Distribution.

Solution:- By def

$$M_x(t) = E(e^{tx})$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{a}^{b} e^{tx} \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \Big|_{a}^{b} e^{tx}$$

$$= \frac{1}{b-a} (e^{tb} - e^{ta})$$

$$\Rightarrow \quad M_x(t) = \frac{e^{tb} - e^{ta}}{b-a}$$

As Required.

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Problem 5) For the P.d.f

$$f(x) = \begin{cases} Kx(1-x) & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine K & Compute $P[\mu - 2\sigma < X < \mu + 2\sigma]$ where μ is Mean & σ is S.D.

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Solution:- As $f(x)$ is p.d.f so,

$$\int_0^1 f(x)\,dx = 1$$

$$\int_0^1 kx(-x+1)\,dx = 1$$

$$-k\int_0^1 (x^2 - x)\,dx = 1$$

$$-k\left[\Big|_0^1 \frac{x^3}{3} - \frac{x^2}{2} \right] = 1$$

$$-k\left[\frac{1}{3} - \frac{1}{2} \right] = 1$$

$$-k\left[\frac{2-3}{6} \right] = 1$$

$$-k\left[\frac{-1}{6} \right] = 1$$

$$k = 6$$

Now Required function will be,

$$f(x) = 6x(1-x) \qquad 0 \le x \le 1$$

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Now, $\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$

$\mu = \int_0^1 x \cdot 6x(1-x)\, dx$

$\mu = 6 \int_0^1 (x^2 - x^3)\, dx$

$\mu = 6 \left| \frac{x^3}{3} - \frac{x^4}{4} \right|_0^1$

$\mu = 6 \left[\frac{1}{3} - \frac{1}{4} \right] = 6 \left[\frac{4-3}{12} \right]$

$\mu = 1/2$

Now,

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$

$= \int_0^1 x^2\, 6x(1-x)\, dx$

$= 6 \int_0^1 (x^3 - x^4)\, dx$

$= 6 \left[\left| \frac{x^4}{4} - \frac{x^5}{5} \right|_0^1 \right]$

$= 6 \left[\frac{1}{4} - \frac{1}{5} \right] = 6 \left[\frac{5-4}{20} \right]$

$= 3/10$

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Now

$$\sigma^2 = E(x^2) - [E(x)]^2$$

$$= 3/10 - \frac{1}{4}$$

$$= \frac{6-5}{20}$$

$$\sigma^2 = 1/20$$

&

$$\sigma = \frac{1}{\sqrt{20}}$$

Now, $\mu - 2\sigma = \frac{1}{2} - \frac{2}{\sqrt{20}} = 0.052$

$$\mu + 2\sigma = \frac{1}{2} + \frac{2}{\sqrt{20}} = 0.948$$

So,

$$P[\mu - 2\sigma < x < \mu + 2\sigma] = \int_{0.052}^{0.948} 6x(1-x)\,dx$$

$$= 6\int_{0.052}^{0.948} (x - x^2)\,dx$$

$$= 6 \left| \frac{x^2}{2} - \frac{x^3}{3} \right|_{0.052}^{0.948}$$

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$$= 6\left\{ \frac{(0.948)^2}{2} - \frac{(0.052)^2}{2} - \frac{(0.948)^3}{3} + \frac{(0.052)^3}{3} \right\}$$

$$= 6\left\{ 0.449 - 1.352\times10^{-3} - 0.283 + 4.686\times10^{-5} \right\}$$

$$= 0.988$$

As Required.

Problem 6) If x is uniformly Distributed b/w $-a$ to $+a$ then find

i) p.d.f $\varphi(x)$

ii) Mean

iii) Show that

$$\mu'_{2n+1} = 0 \quad \& \quad \mu'_{2n} = \frac{a^{2n}}{2n+1}$$

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Solution:-

As $\varphi(x)$ is p.d.f so,

$$\int_{-a}^{a} \varphi(x)\,dx = 1$$

$$\varphi(x) \left| x \right._{-a}^{b} = 1$$

$$\varphi(x)\,(a+a) = 1$$

$$\Rightarrow \varphi(x) = 1/2a \qquad -a \leq x \leq a$$

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ii) Mean = $E(X) = \int_{-\infty}^{\infty} x\, \varphi(x)\, dx$

$= \int_{-a}^{a} x \cdot \frac{1}{2a}\, dx$

$= \frac{1}{2a} \Big|_{-a}^{a} \frac{x^2}{2}$

$= \frac{1}{2a} \left\{ \frac{a^2}{2} - \frac{a^2}{2} \right\}$

$= 0$

iii) $\mu'_{2n+1} = \int_{-\infty}^{\infty} x^{2n+1} f(x)\, dx$

$= \int_{-\infty}^{\infty} x^{2n+1} \varphi(x)\, dx$

$= \int_{-a}^{a} x^{2n+1} \cdot \frac{1}{2a}\, dx$

$= \frac{1}{2a} \Big|_{-a}^{a} \frac{x^{2n+2}}{2n+2}$

$= \frac{1}{2a(2n+2)} \left\{ a^{2n+2} - (-a)^{2n+2} \right\}$

$= \frac{1}{2a(2n+2)} \left\{ a^{2n+2} - a^{2n+2} \right\}$

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$$= \frac{0}{2a(2n+2)}$$

$$= 0$$

As Required.

$$\mu'_{2n} = \int_{-a}^{a} x^{2n} \varphi(x)\,dx$$

$$= \frac{1}{2a}\int_{-a}^{a} x^{2n}\,dx$$

$$= \frac{1}{2a}\left|\frac{x^{2n+1}}{2n+1}\right|_{-a}^{a}$$

$$= \frac{1}{2a(2n+1)}\left(a^{2n+1} + a^{2n+1}\right)$$

$$= \frac{2a^{2n+1}}{2a(2n+1)}$$

$$= \frac{2a \cdot a^{2n}}{2a(2n+1)}$$

$$\mu'_{2n} = \frac{a^{2n}}{(2n+1)}$$

As Required.

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2- GAMA Distribution

A Continous r.v X is said to have Gama Distribution with Parameter $m > 0$ if its P.d.f is given by,

$$f(x) = \frac{1}{\Gamma(m)} \cdot x^{m-1} e^{-x} \qquad 0 \leq x < \infty$$

A Gamma Variable with Parameter "m" is usually denoted by $\gamma(m)$.

*GAMA Function :-

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dx$$

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^{x} \, dx$$

$$\Gamma(n) = (n-1)!$$

$$\Gamma(1/2) = \sqrt{\pi}$$

*

Problem 1) Find Mean & Variance of GAMA Distribution.

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Solution:- By def

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_0^{\infty} x \cdot \frac{1}{\Gamma(m)} x^{m-1} e^{-x}\,dx$$

$$= \frac{1}{\Gamma(m)} \int_0^{\infty} e^{-x} x^{m}\,dx$$

$\Gamma n = (n-1)\Gamma n-1$

$$= \frac{1}{\Gamma(m)} \cdot \Gamma(m+1)$$

$$= \frac{1}{\Gamma(m)} \cdot m\Gamma(m)$$

$$= m$$

$\Rightarrow$ Mean $= m$

Now, $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$$= \int_0^{\infty} x^2 \cdot \frac{1}{\Gamma(m)} x^{m-1} e^{-x}\,dx$$

$$= \frac{1}{\Gamma(m)} \int_0^{\infty} x^{m+1} e^{-x}\,dx$$

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$$= \frac{1}{\Gamma(m)} \cdot \Gamma(m+2)$$

$$= \frac{1}{\Gamma(m)} \cdot m+1 \; \Gamma(m+1)$$

$$= \frac{1}{\Gamma(m)} \cdot \underline{(m+1)\, m\, \Gamma(m)}$$

$$E(x^2) = m(m+1)$$

So,

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= m(m+1) - m^2$$

$$= m^2 + m - m^2$$

$$Var(X) = m$$

As Required.

Problem 2) Find Moment Generating Function of Gama Distribution & using it find moments about mean & then find β-coefficients.

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Solution:- By defination

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_0^{\infty} e^{tx} \cdot \frac{1}{\Gamma(m)} e^{-x} x^{m-1} dx$$

$$= \frac{1}{\Gamma(m)} \int_0^{\infty} e^{-x(1-t)} x^{m-1} dx$$

Put $u = x(1-t)$

$du = dx(1-t)$

$dx = \frac{du}{1-t}$

$x = \frac{u}{1-t}$

As $x \longrightarrow 0$ then $u \longrightarrow 0$

As $x \longrightarrow \infty$ then $u \longrightarrow \infty$

So, we have

$$M_x(t) = \frac{1}{\Gamma(m)} \int_0^{\infty} e^{-u} \left(\frac{u}{1-t}\right)^{m-1} \frac{du}{(1-t)}$$

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$$= \frac{1}{\Gamma(m)} \int_0^\infty e^{-u} \frac{(u)^{m-1}}{(1-t)^m} du$$

$$= \frac{1}{\Gamma(m)\,(1-t)^m} \int_0^\infty e^{-u} u^{m-1} du$$

$$= \frac{1}{\Gamma(m)\,(1-t)^m} \cdot \Gamma(m)$$

$$= \frac{1}{(1-t)^m} = (1-t)^{-m}$$

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Now,

$$M_x(t) = (1-t)^{-m}$$

which is Required m.g.f.

Now,

$$M_x(t) = 1 + mt + \frac{m(m+1)}{2!} t^2 + \frac{m(m+1)(m+2)}{3!} t^3$$

Now comparing the co-efficients of $\frac{t^r}{r!}$ we get,

$\mu_1' = m$

$\mu_2' = m(m+1)$

$\mu_3' = m(m+1)(m+2)$

$\mu_4' = m(m+1)(m+2)(m+3)$

Now Moments About Mean,

$\mu_1 = 0$

$\mu_2 = \mu_2' - (\mu_1')^2$

$= m^2 + m - m^2$

$= m$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

$= m(m+1)(m+2) - 3(m(m+1))(m) + 2m^3$

$= m(m^2+3m+2) - 3m^2(m+1) + 2m^3$

$= m^3 + 3m^2 + 2m - 3m^3 - 3m^2 + 2m^3$

$\mu_3 = 2m$

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$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$\mu_4 = m(m+1)(m+2)(m+3) - 4(m^2(m+1))(m+2)$
$+ 6(m(m+1))m^2 - 3m^4$

$\mu_4 = (m^2+m)(m^2+5m+6) - 4(m^3+m^2)(m+2)$
$+ 6m^3(m+1) - 3m^4$

$\mu_4 = m^4 + 5m^3 + 6m^2 + m^3 + 5m^2 + 6m$
$- 4m^4 - 4m^3 - 8m^3 - 8m^2 + 6m^4 + 6m^3$
$- 3m^4$

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$\mu_4 = 3m^2 + 6m$

$\mu_4 = 3m(m+2)$

Now,

$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(2m)^2}{m^3} \Rightarrow \frac{4}{m}$

$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3m(m+2)}{m^2} = \frac{3(m+2)}{m}$

As Required.

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Problem 3) Find Cumulant Generating Function for Gamma Distribution & using it to find 1st four cumulants.

Solution:- As we know

$$K(t) = \log_e M_x(t)$$

$$K(t) = \log_e (1-t)^{-m}$$

$$K(t) = -m \log(1-t)$$

$$K(t) = m\left[t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4} + \cdots\right]$$

$$K(t) = mt + \frac{mt^2}{2!} + 2m\frac{t^3}{3!} + 6m\frac{t^4}{4!} + \cdots$$

$$K_1(t) + K_2\left(\frac{t^2}{2!}\right) + K_3\left(\frac{t^3}{3!}\right) + K_4\left(\frac{t^4}{4!}\right) + \cdots$$

$$= mt + \frac{mt^2}{2!} + 2m\frac{t^3}{3!} + 6m\frac{t^4}{4!}$$

By Comparing Co-efficient we have,

$K_1 = m$

$K_2 = m$

$K_3 = 2m$

$K_4 = 6m$

As Required.

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Problem 4) Show that the function

$$f(x) = \frac{e^{-x} x^{m-1}}{\Gamma(m)} \quad 0 < x < \infty$$

is p.d.f.

Solution:- As we know that function will be p.d.f if

$$\int_0^\infty f(x)\,dx = 1$$

$$R.H.S = \int_0^\infty f(x)\,dx$$

$$= \int_0^\infty \frac{e^{-x} x^{m-1}}{\Gamma(m)}\,dx$$

$$= \frac{1}{\Gamma(m)} \cdot \Gamma(m)$$

$$= 1 = L.H.S$$

As Required.

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"Reproductive Property of Gama Variates"

let X & Y be independent γ- Variates with Parameters l & m then $X+Y$ is also γ variate with Parameter $(l+m)$.

Proof:- $M_{x+y}(t) = E\left[e^{t(x+y)}\right]$

$= E\left[e^{tx} \cdot e^{ty}\right]$

$= E(e^{tx}) \cdot E(e^{ty})$ $\because$ X & Y are independent.

Now,

$M_x(t) = (1-t)^{-l}$

$M_y(t) = (1-t)^{-m}$

$M_{x+y}(t) = (1-t)^{-l} \cdot (1-t)^{-m}$

$= (1-t)^{-(l+m)}$

which shows that $X+Y$ is γ variates with Parameter $(l+m)$.

As Required.

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HARMONIC MEAN of GAMA Distribution

Let Harmonic Mean be H

$$E(X) = \text{Mean} = \frac{\sum x}{n}$$

$$H = H.M = \frac{n}{\sum\left(\frac{1}{x}\right)}$$

$$\frac{1}{H} = \frac{\sum\left(\frac{1}{x}\right)}{n} = E\left(\frac{1}{x}\right)$$

$$\frac{1}{H} = E\left(\frac{1}{x}\right)$$

$$= \int_0^\infty \frac{1}{x} f(x)\, dx$$

$$= \int_0^\infty \frac{1}{x} \cdot \frac{1}{\Gamma(m)} e^{-x} x^{m-1}\, dx$$

$$\frac{1}{H} = \frac{1}{\Gamma(m)} \int_0^\infty x^{m-2} e^{-x}\, dx$$

$$\frac{1}{H} = \frac{1}{\Gamma(m)} \Gamma(m-1)$$

$$= \frac{1}{(m-1)\Gamma(m-1)} \cdot \Gamma(m-1)$$

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$$\frac{1}{H} = \frac{1}{m-1}$$

$$\Rightarrow \quad H = m-1$$

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Problem:- Find Mode of Gamma Distribution.

Solution:- The Mode of Gamma Distribution is that value of x for which p.d.f is max.

As $$f(x) = \frac{1}{\Gamma(m)} e^{-x} x^{m-1}$$

$$f'(x) = \frac{1}{\Gamma(m)} \left[-e^{-x} x^{m-1} + (m-1) x^{m-2} e^{-x} \right]$$

$$= \frac{1}{\Gamma(m)} x^{m-1} e^{-x} \left[\frac{m-1}{x} - 1 \right]$$

$$f'(x) = \frac{1}{\Gamma(m)} x^{m-1} e^{-x} \left[\frac{m-1-x}{x} \right]$$

Put $f'(x) = 0$

$$\frac{1}{\Gamma(m)} x^{m-1} e^{-x} \left[\frac{m-1-x}{x} \right] = 0$$

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As $\frac{1}{\Gamma(m)} x^{m-1} e^{-x} \neq 0$ ∵ p.d.f

$\Rightarrow \frac{m-1-x}{x} = 0$

$\Rightarrow m-1-x = 0$

$\Rightarrow x = m-1$

Now,

$$f''(x) = \frac{1}{\Gamma(m)}\Big[-(m-1)e^{-x}x^{m-3} + (m-1-x)e^{-x}(m-1)x^{m-3} - x^{m-2}(m-1-x)e^{-x}\Big]$$

Put $x = m-1$

$$f''(x) = \frac{-1}{\Gamma(m)}(m-1)e^{-(m-1)}(m-1)^{m-3} < 0$$

$\Rightarrow x = m-1$ is Mode of Gamma Distribution.

As Required.

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Theorem:- The sum of n-independent γ-Variates with Parameters m_i ($i = 1, 2, \ldots, n$) is a γ-Variate with Parameter $m_1 + m_2 + \cdots + m_n$

$$= \sum_{i=1}^{n} m_i$$

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Proof:- Now,

$$M_{x_1+x_2+\cdots+x_n}(t) = E\left[e^{t(x_1+x_2+\cdots+x_n)}\right]$$

$$= E\left[e^{tx_1} \cdot e^{tx_2} \cdot e^{tx_3} \cdots e^{tx_n}\right]$$

$$= E(e^{tx_1}) \cdot E(e^{tx_2}) \cdot E(e^{tx_3}) \cdots E(e^{tx_n})$$

$$\therefore E(XY) = E(X)E(Y)$$

$$= (1-t)^{-m_1} \cdot (1-t)^{-m_2} \cdots (1-t)^{-m_n}$$

$$= (1-t)^{-m_1-m_2-m_3-\cdots-m_n}$$

$$= (1-t)^{-(m_1+m_2+m_3+\cdots+m_n)}$$

which is Required m.g.f of γ-Variates with Parameter $\sum_{i=1}^{n} m_i$.

As Required.

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Problem :- Determine y_0 for $f(x) = y_0 x^{n-1} e^{-x}$
$0 \leq x < \infty$

be a probability Density Function & find β_1 & β_2 for this Distribution & also shows that

$$2\mu_4 \sigma^2 - 3\mu_3^2 - 6\sigma^6 = 0$$

Solution:- As Given function is P.d.f So,

$$\int_0^\infty y_0 \, x^{n-1} e^{-x} dx = 1$$

$$y_0 \int_0^\infty x^{n-1} e^{-x} dx = 1$$

$$y_0 = \frac{1}{\Gamma(n)} \quad \text{therefor} \quad \int_0^\infty x^{n-1} e^{-x} = \Gamma(n)$$

So, Required function is,

$$f(x) = \frac{1}{\Gamma(n)} e^{-x} x^{n-1}$$

Now,

$$\mu_r' = E(X^r) = \int_0^\infty x^r f(x) dx$$

$$\mu_r' = \int_0^\infty x^r \cdot \frac{1}{\Gamma(n)} e^{-x} x^{n-1} dx$$

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$$\mu_r' = \frac{1}{\Gamma(n)} \int_0^\infty e^{-x} x^{(n+r)-1} \, dx$$

$$\mu_r' = \frac{1}{\Gamma(n)} \cdot \Gamma(n+r)$$

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put $r = 1$

$$\mu_1' = \frac{\Gamma(n+1)}{\Gamma(n)} = \frac{n\,\Gamma(n)}{\Gamma(n)}$$

$$\mu_1' = n$$

put $r = 2$

$$\mu_2' = \frac{\Gamma(n+2)}{\Gamma(n)} = \frac{n+1 \cdot n \cdot \Gamma(n)}{\Gamma(n)}$$

$$\mu_2' = n(n+1)$$

put $r = 3$

$$\mu_3' = \frac{\Gamma(n+3)}{\Gamma(n)} = \frac{(n+2)(n+1)(n)\,\Gamma(n)}{\Gamma(n)}$$

$$\mu_3' = n(n+1)(n+2)$$

put $r = 4$

$$\mu_4' = \frac{\Gamma(n+4)}{\Gamma(n)} = \frac{(n+3)(n+2)(n+1)\,n\,\Gamma(n)}{\Gamma(n)}$$

$$\mu_4' = n(n+1)(n+2)(n+3)$$

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Now,

$\mu_1 = 0$

$\mu_2 = \mu_2' - (\mu_1')^2$

$= n^2 + n - n^2$

$= n$

$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2\mu_1'^3$

$\mu_3 = n(n+1)(n+2) - 3(n^2(n+1)) + 2n^3$

$\mu_3 = n^3 + 3n^2 + 2n - 3n^3 - 3n^2 + 2n^3$

$\mu_3 = 2n$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$\mu_4 = n(n+1)(n+2)(n+3) - 4(n(n+1)(n+2))n$
$+ 6\,n(n+1)n^2 - 3n^4$

$\mu_4 = (n^2+n)(n^2+5n+6) - 4n^2(n^2+3n+2)$
$+ 6n^3(n+1) - 3n^4$

$\mu_4 = n^4 + 5n^3 + 6n^2 + n^3 + 5n^2 + 6n - 4n^4 - 12n^3 - 8n^2$
$+ 6n^4 + 6n^3 - 3n^4$

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$\mu_4 = 3n^2 + 6n$

Now

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(2n)^2}{(n)^3} \Rightarrow \frac{4n^2}{n^3}$$

$$\beta_1 = \frac{4}{n}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3n^2+6n}{(n)^2}$$

$$\beta_2 = \frac{3n^2+6n}{n^2} = \left(3 + \frac{6}{n}\right)$$

Now we have to show that

$$2\mu_4\sigma^2 - 3\mu_3^2 - 6\sigma^6 = 0$$

$$L.H.S = 2\mu_4\sigma^2 - 3\mu_3^2 - 6(\sigma^2)^3$$

$$= 2(3n^2+6n)n - 3(2n)^2 - 6(n)^3$$

$$= 6n^3 + 12n^2 - 12n^2 - 6n^3$$

$$= 0$$

$$= R.H.S$$

As Required.

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β - Function

The β function for any two +ve numbers m, n denoted by β(m,n) & is defined by,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad m,n>0$$

Properties of β-Function :-

P1) $\beta(1,1) = 1$

As,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Put $m = n = 1$

$$\beta(1,1) = \int_0^1 x^0(1-x)^0\,dx$$

$$\beta(1,1) = \int_0^1 dx = \Big|_0^1 x$$

$$\beta(1,1) = 1$$

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P2) β-function is symmeterical w.r.t parameters m & n. i.e,

$$\beta(m,n) = \beta(n,m)$$

As,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Put $y = 1-x \Rightarrow dx = -dy$

As $x = 0 \Rightarrow y = 1$

As $x = 1 \Rightarrow y = 0$

$$\beta(m,n) = \int_1^0 (1-y)^{m-1}(y)^{n-1}(-dy)$$

$$= \int_0^1 (1-y)^{m-1} y^{n-1} dy$$

$$\beta(m,n) = \beta(n,m).$$

P3) $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

As $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

Put $x = \sin^2\theta$, $1-x = \cos^2\theta$

$dx = 2\sin\theta\cos\theta \, d\theta$

$x = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \theta = \pi/2$

$$\beta(m,n) = \int_0^{\pi/2} \sin^{2(m-1)}\theta \cos^{2(n-1)}\theta \; 2\sin\theta.\cos\theta \, d\theta$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

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P4) $\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$

As $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = 2\int_0^{\pi/2} \sin^0\theta \cos^0\theta \, d\theta$$

$$= 2 \Big|_0^{\pi/2} \theta$$

$$= 2\left(\pi/2 - 0\right)$$

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$$

P5) $\beta(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} \, dx$

Mr. Zahid this step is wrong

As, $\beta(m,n) = \int_0^{1} \frac{x^{m-1}}{(1+x)^{m+n}} \, dx.$

Put $x = \frac{1}{1+z}$

$$dx = \frac{-1}{(1+z)^2} \, dz$$

$$x = 1 \Rightarrow z = 0$$

$$x = 0 \Rightarrow z = \infty$$

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$$\beta(m,n) = \int_{\infty}^{0} \left(\frac{1}{1+z}\right)^{m-1} \left(\frac{z}{1+z}\right)^{n-1} \left(\frac{-1}{(1+z)^2}\right) dz$$

$$\beta(m,n) = \int_{0}^{\infty} \frac{z^{n-1}}{(1+z)^{n+m}} dz$$

$$\beta(m,n) = \int_{0}^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

As Required.

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Relationship B/w Gamma & Beta Function.

Prove that

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \qquad m>0,\ n>0$$

Proof:- Let $\Gamma(n) = I_1 = \int_0^\infty e^{-x} x^{n-1} dx$

$\Gamma(m) = I_2 = \int_0^\infty e^{-y} y^{m-1} dy$

$$I_1 I_2 = \int_0^\infty \int_0^\infty e^{-(x+y)} x^{n-1} y^{m-1} dx\,dy$$

Put $x = uz$, $y = u(1-z)$

$x+y = u$, $0 \le x < \infty$

$0 \le y < \infty$

$0 \le u < \infty$, $0 \le z \le 1$

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$$J = \frac{\partial(x,y)}{\partial(u,z)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial z \\ \frac{\partial y}{\partial u} & \partial y/\partial z \end{vmatrix}$$

$$J = \begin{vmatrix} z & u \\ 1-z & -u \end{vmatrix}$$

$$J = -zu - u + zu = -u$$

$$dxdy = |J| du dz$$

$$dxdy = u du dz$$

So,

$$I_1 I_2 = \int_0^\infty \int_0^1 e^{-u} u^{n-1} z^{n-1} u^{m-1} (1-z)^{m-1} u \, du dz$$

$$= \int_0^\infty \int_0^1 e^{-u} z^{n-1} u^{(n+m)-1} (1-z)^{m-1} du dz$$

$$I_1 I_2 = \int_0^\infty e^{-u} u^{(n+m)-1} du \int_0^1 z^{n-1} (1-z)^{m-1} dz$$

$$\Gamma(n)\Gamma(m) = \Gamma(m+n) \, \beta(m,n)$$

$$\Rightarrow \beta(m,n) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(m+n)}$$

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Problem:- Prove that

$$\Gamma\tfrac{1}{2} = \sqrt{\pi}$$

Proof:- As we know that

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Put $m = \tfrac{1}{2}$ &

$n = \tfrac{1}{2}$

$$\beta(\tfrac{1}{2}, \tfrac{1}{2}) = \frac{\Gamma\tfrac{1}{2}\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}+\tfrac{1}{2})}$$

$$\beta(\tfrac{1}{2}, \tfrac{1}{2}) = \left[\Gamma\tfrac{1}{2}\right]^2$$

As we know $\beta(\tfrac{1}{2}, \tfrac{1}{2}) = \pi$

So,

$$\Gamma\tfrac{1}{2} = \sqrt{\pi}$$

As Required.

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Beta Distribution of 1st Kind.

A Continous r.v X is said to have a Beta Distribution of 1st kind of two Parameters m,n if its p.d.f is defined by,

$$f(x) = \begin{cases} \frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1} & 0 \leq x \leq 1 \\ 0 & \text{Elsewise} \end{cases}$$

This Distribution is also called β_1- distribution.

Problem 1) Show that total Area under the β_1 curve is 1.

Solution:- Now Consider

$$\int_0^1 \frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1}\,dx = \frac{1}{\beta(m,n)} \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \frac{1}{\beta(m,n)} \cdot \beta(m,n)$$

$$= 1$$

As Required.

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Problem 2) Find Mean & Var of β_2-distribution.

Solution:- As we know that

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_0^1 x \cdot \frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1}\, dx$$

$$= \frac{1}{\beta(m,n)} \int_0^1 x^{(m+1)-1}(1-x)^{n-1}\, dx$$

$$= \frac{1}{\beta(m,n)} \cdot \beta(m+1,n)$$

$$= \frac{\Gamma(m+1)\,\Gamma(n)}{\Gamma(m+n+1)} \cdot \frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)}$$

$$= \frac{m\,\Gamma(m)\,\Gamma n}{m+n\ \Gamma(m+n)} \cdot \frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)}$$

$$= \frac{m}{m+n}$$

$$\Rightarrow \text{Mean} = \frac{m}{m+n}$$

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Now,

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_0^1 x^2 \cdot \frac{1}{\beta(m,n)} x^{m-1} (1-x)^{n-1} dx$$

$$= \frac{1}{\beta(m,n)} \int_0^1 x^{(m+2)-1} (1-x)^{n-1} dx$$

$$= \frac{1}{\beta(m,n)} \cdot \beta(m+2, n)$$

$$= \frac{\Gamma(m+2)\,\Gamma(n)}{\Gamma(m+n+2)} \cdot \frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)}$$

$$= \frac{(m+1)\, m\, \Gamma(m)\, \Gamma(n)}{(m+n+1)(m+n)\,\Gamma(m+n)} \cdot \frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)}$$

$$E(X^2) = \frac{m(m+1)}{(m+n)(m+n+1)}$$

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Now,

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{m(m+1)}{(m+n)(m+n+1)} - \left[\frac{m}{m+n}\right]^2$$

$$= \frac{m}{m+n}\left[\frac{m+1}{m+n+1} - \frac{m}{m+n}\right]$$

$$= \frac{m}{m+n}\left[\frac{m^2+m+mn+n-m^2-mn-m}{(m+n)(m+n+1)}\right]$$

$$= \frac{mn}{(m+n)^2(m+n+1)}$$

As Required.

Problem 3) Find H.M of β_1- distribution.

Solution:- As we know that

$$\frac{1}{H} = E(1/x) = \int_{-\infty}^{\infty} \frac{1}{x} \cdot f(x)\,dx$$

$$\frac{1}{H} = \int_0^1 \frac{1}{x} \cdot \frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1}\,dx$$

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$$\frac{1}{H} = \frac{1}{\beta(m,n)} \int_0^1 x^{(m-1)-1}(1-x)^{n-1}\,dx$$

$$\frac{1}{H} = \frac{1}{\beta(m,n)} \cdot \beta(m-1,n)$$

$$= \frac{\Gamma(m-1)\,\Gamma(n)}{\Gamma(m+n-1)} \cdot \frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)}$$

$$= \frac{\Gamma(m-1)\,\Gamma(n)}{\Gamma(m+n-1)} \cdot \frac{(m+n-1)\,\Gamma(m+n-1)}{(m-1)\,\Gamma(m-1)\,\Gamma(n)}$$

$$= \frac{m+n-1}{m-1}$$

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$$\Rightarrow H = \frac{m-1}{m+n-1}$$

As Required.

Problem 4) Find Mode of β_1- distribution

Solution:- Mode is that value of Variate x for which $f(x)$ is maximum.

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Now,

$$f(x) = \frac{1}{\beta(m,n)} x^{m-1} (1-x)^{n-1} \quad m,n>0$$

$$0 \le x \le 1$$

$$f'(x) = \frac{1}{\beta(m,n)} \left[(m-1) x^{m-2} (1-x)^{n-1} + x^{m-1} (n-1)(1-x)^{n-2} (-1) \right]$$

$$= \frac{1}{\beta(m,n)} x^{m-2} (1-x)^{n-2} \left[-x(n-1) + (m-1)(1-x) \right]$$

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As $f'(x) = 0$

$$\Rightarrow -xn + x + m - mx - 1 + x = 0$$

$$2x - (m+n)x = 1 - m$$

$$x[2 - m - n] = 1 - m$$

$$x = \frac{1-m}{2-m-n} = \frac{m-1}{m+n-2}$$

It can be shows that

$$f''(x) < 0 \text{ at } x = \frac{m-1}{m+n-2}$$

Hence,

$$\text{Mode} = \frac{m-1}{m+n-2}$$

As Required.

Problem 5) Find 1st four moments about Mean & calculate the co-efficient of Skewness & Kurtosis.

Solution:- By defination

$$\mu_r' = E(x^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx$$

$$= \int_0^1 x^r \frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1}\,dx$$

$$\mu_r' = \frac{1}{\beta(m,n)} \int_0^1 x^{(m+r)-1}(1-x)^{n-1}\,dx$$

$$\mu_r' = \frac{1}{\beta(m,n)} \cdot \beta(m+r, n)$$

$$\mu_r' = \frac{\Gamma(m+r)\,\Gamma(n)}{\Gamma(m+r+n)} \cdot \frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)}$$

Put $r = 1$

$$\mu_1' = \frac{\Gamma(m+1)\,\Gamma(n)}{\Gamma(m+n+1)} \cdot \frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)}$$

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$$\mu_1' = \frac{m\,\Gamma(m)}{(m+n)\,\Gamma(m+n)} \cdot \frac{\Gamma(m+n)}{\Gamma(m)}$$

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$$\mu_1' = \frac{m}{m+n}$$

put $r = 2$

$$\mu_2' = \frac{\Gamma(m+2)}{\Gamma(m+n+2)} \cdot \frac{\Gamma(m+n)}{\Gamma(m)}$$

$$\mu_2' = \frac{m(m+1)\,\Gamma(m)}{(m+n+1)(m+n)\,\Gamma(m+n)} \cdot \frac{\Gamma(m+n)}{\Gamma(m)}$$

$$\mu_2' = \frac{m(m+1)}{(m+n)(m+n+1)}$$

put $r = 3$

$$\mu_3' = \frac{\Gamma(m+3)}{\Gamma(m+n+3)} \cdot \frac{\Gamma(m+n)}{\Gamma(m)}$$

$$\mu_3' = \frac{(m+2)(m+1)(m)\,\Gamma(m)}{(m+n+2)(m+n+1)(m+n)\,\Gamma(m+n)} \cdot \frac{\Gamma(m+n)}{\Gamma(m)}$$

$$\mu_3' = \frac{m(m+1)(m+2)}{(m+n)(m+n+1)(m+n+2)}$$

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$$\mu_4' = \frac{\Gamma(m+4)}{\Gamma(m+n+4)} \cdot \frac{\Gamma(m+n)}{\Gamma(m)}$$

$$\mu_4' = \frac{(m+3)(m+2)(m+1)\,m\,\Gamma(m)}{(m+n+3)(m+n+2)(m+n+1)(m+n)\,\Gamma(m+n)} \cdot \frac{\Gamma(m+n)}{\Gamma(m)}$$

$$\Rightarrow \mu_4' = \frac{m(m+1)(m+2)(m+3)}{(m+n)(m+n+1)(m+n+2)(m+n+3)}$$

Now moments about Mean.

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$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - \mu_1'^2$$

$$= \frac{m(m+1)}{(m+n)(m+n+1)} - \frac{m^2}{(m+n)^2}$$

$$= \frac{m}{m+n}\left[\frac{(m+1)(m+n) - m(m+n+1)}{(m+n)(m+n+1)}\right]$$

$$= \frac{m}{(m+n)^2(m+n+1)}\left[m^2+mn+m+n-m^2-mn-m\right]$$

$$\mu_2 = \frac{mn}{(m+n)^2(m+n+1)}$$

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2\mu_1'^3$$

By putting values we get

$$\mu_3 = \frac{2mn(m-n)}{(m+n)^2(m+n+1)(m+n+2)}$$

$$\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6\mu_1'^2\mu_2' - 3\mu_1'^4$$

we get,

$$\mu_4 = \frac{3mn\left[mn(m+n-6)+2(m+n)^2\right]}{(m+n)^4(m+n+1)(m+n+2)(m+n+3)}$$

Now,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{4(m-n)^2(m+n+1)}{mn(m+n+2)^2}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3(m+n+1)\left[mn(m+n-6)+2(m+n)^2\right]}{mn(m+n+2)(m+n+3)}$$

As Required.

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β Distribution of 2nd Kind

A Continious r.v X is said to have a β Distribution of 2nd order if its p.d.f is defined by,

$$f(x) = \begin{cases} \frac{1}{\beta(m,n)} \frac{x^{m-1}}{(1+x)^{m+n}} & 0 \leq x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Problem:- Show that Area under the β_2 curve is unity.

Now Consider,

$$\int_0^\infty \frac{1}{\beta(m,n)} \frac{x^{m-1}}{(1+x)^{m+n}} dx = \frac{1}{\beta(m,n)} \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

put $1+x = \frac{1}{y}$

$$x = \frac{1-y}{y}$$

$$dx = \frac{-1}{y^2} dy$$

As $x = 0 \Rightarrow y = 1$

As $x = \infty \Rightarrow y = 0$

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$$= \frac{1}{\beta(m,n)} \int_1^0 \left(\frac{1-y}{y}\right)^{m-1} (y)^{m+n} \left(-\frac{dy}{y^2}\right)$$

$$= \frac{1}{\beta(m,n)} \int_0^1 y^{n-1} (1-y)^{m-1} \, dy$$

$$= \frac{1}{\beta(m,n)} \cdot \beta(m,n)$$

$$= 1$$ As Required.

P2) Find Mean & Var of β_2 – Distribution.

Solution:- As we know that

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x \, f(x) \, dx$$

$$= \int_0^{\infty} x \cdot \frac{1}{\beta(m,n)} \cdot \frac{x^{m-1}}{(1+x)^{m+n}} \, dx$$

$$= \frac{1}{\beta(m,n)} \int_0^{\infty} \frac{x^{(m+1)-1}}{(1+x)^{(m+1)+(n-1)}} \, dx$$

$$= \frac{1}{\beta(m,n)} \cdot \beta(m+1, n-1)$$

$$= \frac{\Gamma(m+1) \cdot \Gamma(m-1)}{\Gamma(m+n)} \cdot \frac{\Gamma(m+n)}{\Gamma(m) \, \Gamma(n)}$$

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$$= \frac{m \,\Gamma(m)\, \Gamma(n-1)}{\Gamma(m)\; n-1\; \Gamma(n-1)}$$

$$\text{Mean} = \frac{m}{n-1}$$

which is Mean of β_2 - Distribution.

Now,

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_0^{\infty} x^2 \cdot \frac{1}{\beta(m,n)} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

$$= \frac{1}{\beta(m,n)} \int_0^{\infty} \frac{x^{(m+2)-1}}{(1+x)^{(m+2)+(n-2)}}\, dx$$

$$= \frac{1}{\beta(m,n)}\; \beta(m+2, n-2)$$

$$= \frac{\Gamma(m+2)\;\Gamma(n-2)}{\Gamma(m+n)} \cdot \frac{\Gamma(m+n)}{\Gamma(m)\;\Gamma(n)}$$

$$= \frac{(m+1)\, m\, \Gamma(m)\; \Gamma(n-2)}{\Gamma(m)\; (n-1)(n-2)\,\Gamma(n-2)}$$

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$$E(x^2) = \frac{m(m+1)}{(n-1)(n-2)}$$

Now,

$$Var(x) = E(x^2) - [E(x)]^2$$

$$= \frac{m(m+1)}{(n-1)(n-2)} - \left[\frac{m}{n-1}\right]^2$$

$$= \frac{m}{n-1}\left[\frac{m+1}{n-2} - \frac{m}{n-1}\right]$$

$$= \frac{m}{n-1}\left[\frac{(m+1)(n-1) - m(n-2)}{(n-1)(n-2)}\right]$$

$$= \frac{m}{(n-1)^2(n-2)}\left[\cancel{mn} - m + n - 1 - \cancel{mn} + 2m\right]$$

$$= \frac{m(m+n-1)}{(n-1)^2(n-2)}$$

$$\Rightarrow \quad Var(x) = \frac{m(m+n-1)}{(n-1)^2(n-2)}$$

As Required.

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Problem 3) Find H.M of β_2 - Distribution.

Solution:- As we know that

$$\frac{1}{H} = E\left(\frac{1}{X}\right) = \int_{-\infty}^{\infty} \frac{1}{x} f(x)\,dx$$

$$= \int_0^{\infty} \frac{1}{x} \cdot \frac{1}{\beta(m,n)} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$= \frac{1}{\beta(m,n)} \int_0^{\infty} \frac{x^{(m-1)-1}}{(1+x)^{(m-1)+(n+1)}}\,dx$$

$$= \frac{1}{\beta(m,n)} \cdot \beta(m-1, n+1)$$

$$= \frac{\Gamma(m-1)\,\Gamma(n+1)}{\Gamma(m+n)} \cdot \frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)}$$

$$= \frac{\Gamma(m-1)\; n\,\Gamma(n)}{m-1\,\Gamma(m-1)\,\Gamma(n)}$$

$$= \frac{n}{m-1} \Rightarrow H = \frac{m-1}{n}$$

As Required.

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Problem 4) Find Mode of β_2 - Distribution.

Solution:- Mode is the that value of variate x for which f(x) is maximum.

$$f(x) = \frac{1}{\beta(m,n)} x^{m-1}(1+x)^{-(m+n)}$$

$$f'(x) = \frac{1}{\beta(m,n)}\left[(m-1)x^{m-2}(1+x)^{-(m+n)} + x^{m-1} - (m+n)(1+x)^{-(m+n+1)}\right]$$

$$= \frac{1}{\beta(m,n)} x^{m-2}(1+x)^{-(m+n+1)}\left[(m-1)(1+x) - (m+n)x\right]$$

put $f'(x) = 0$

$$\Rightarrow \quad m + mx - 1 - x - mx - nx = 0$$

$$m - 1 = x(n+1)$$

$$\Rightarrow \quad x = \frac{m-1}{n+1}$$

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Now It is Easy to see

$$f''(x) < 0 \quad \text{at } x = \frac{m-1}{n+1}$$

$$\Rightarrow \quad \text{Mode} = \frac{m-1}{n+1} \quad \text{As Required.}$$

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Problem 5) Find 1st 4-Moments about origin of β_2- Distribution.

Solution:- Now consider

$$\mu_r' = E(x^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx$$

$$= \int_0^{\infty} x^r \frac{1}{\beta(m,n)} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$= \frac{1}{\beta(m,n)} \int_0^{\infty} \frac{x^{(m+r)-1}}{(1+x)^{(m+r)+(n-r)}}\,dx$$

$$= \frac{1}{\beta(m,n)} \cdot \beta(m+r,\ n-r)$$

$$= \frac{\Gamma(m+r)\,\Gamma(n-r)}{\Gamma(m+n)} \cdot \frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)}$$

$$\mu_r' = \frac{\Gamma(m+r)\,\Gamma(n-r)}{\Gamma(m)\,\Gamma(n)}$$

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put $r = 1$

$$\mu_1' = \frac{\Gamma(m+1)\,\Gamma(n-1)}{\Gamma(m)\,\Gamma(n)}$$

$$\mu_1' = \frac{m\,\Gamma(m)\,\Gamma(n-1)}{\Gamma(m)\,(n-1)\,\Gamma(n-1)}$$

$$\mu_1' = \frac{m}{n-1}$$

put $r = 2$

$$\mu_2' = \frac{\Gamma(m+2)\,\Gamma(n-2)}{\Gamma(m)\,\Gamma(n)}$$

$$\mu_2' = \frac{(m+1)\,m\,\Gamma(m)\,\Gamma(n-2)}{\Gamma(m)\,(n-1)\,(n-2)\,\Gamma(n-2)}$$

$$\mu_2' = \frac{m(m+1)}{(n-1)(n-2)}$$

put $r = 3$

$$\mu_3' = \frac{\Gamma(m+3)\,\Gamma(n-3)}{\Gamma(m)\,\Gamma(n)}$$

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$$\mu_3' = \frac{(m+2)(m+1)\, m\, \Gamma(m)\, \Gamma(n-3)}{\Gamma(m)\,(n-1)(n-2)(n-3)\,\Gamma(n-3)}$$

$$\mu_3' = \frac{m(m+1)(m+2)}{(n-1)(n-2)(n-3)}$$

put $r = 4$

$$\mu_4' = \frac{\Gamma(m+4)\, \Gamma(n-4)}{\Gamma(m)\, \Gamma(n)}$$

$$\mu_4' = \frac{(m+3)(m+2)(m+1)\, m\, \Gamma(m)\, \Gamma(n-4)}{\Gamma(m)\,(n-1)(n-2)(n-3)(n-4)\,\Gamma(n-4)}$$

$$\mu_4' = \frac{m(m+1)(m+2)(m+3)}{(n-1)(n-2)(n-3)(n-4)}$$

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As Required

"Exponential Distribution"

The random variable X is said to have an Exponential distribution if its p.d.f is given by;

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & ; x > 0 \\ 0 & \text{Elsewhere} \end{cases}$$

OR

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & ; x > 0 \\ 0 & \text{Elsewhere} \end{cases}$$

where α is the Parameter.

Problem 1) Show that Area under the Exponential curve is unity.

Solution:- As we know that

$$\text{Area} = \int_{-\infty}^{\infty} f(x)\,dx$$

$$= \alpha \int_0^{\infty} e^{-\alpha x}\,dx$$

$$= \alpha \left| \frac{e^{-\alpha x}}{-\alpha} \right|_0^{\infty}$$

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$$= -\{0 - 1\}$$

$$= 1$$

$\Rightarrow$ Area $= 1$ As Required.

Problem 2) Find Mean & Variance of the Exponential Distribution.

Solution:-

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{0}^{\infty} x \cdot \alpha e^{-\alpha x}\,dx$$

$$= \alpha \int_{0}^{\infty} x e^{-\alpha x}\,dx$$

$$= \alpha \left[\left| x \frac{e^{-\alpha x}}{-\alpha} \right|_{0}^{\infty} + \frac{1}{\alpha} \int_{0}^{\infty} e^{-\alpha x}\,dx \right]$$

$$= \int_{0}^{\infty} e^{-\alpha x}\,dx$$

$$= \frac{-1}{\alpha} \left| e^{-\alpha x} \right|_{0}^{\infty} \Rightarrow \frac{-1}{\alpha}\{0 - 1\}$$

$\Rightarrow$ Mean $= \dfrac{1}{\alpha}$

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$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_0^{\infty} x^2 \alpha e^{-\alpha x}\,dx$$

$$= \alpha \int_0^{\infty} x^2 e^{-\alpha x}\,dx$$

$$= \alpha \left[\left| x^2 \frac{e^{-\alpha x}}{-\alpha} \right|_0^{\infty} + \frac{1}{\alpha} \int_0^{\infty} e^{-\alpha x}(2x)\,dx \right]$$

$$= 2 \int_0^{\infty} e^{-\alpha x}\, x\,dx$$

$$= 2 \left[\left| x \frac{e^{-\alpha x}}{-\alpha} \right|_0^{\infty} + \frac{1}{\alpha} \int_0^{\infty} e^{-\alpha x}\,dx \right]$$

$$= 2 \left[0 + \frac{1}{\alpha} \left| \frac{e^{-\alpha x}}{-\alpha} \right|_0^{\infty} \right]$$

$$= -\frac{2}{\alpha^2} \left[0 - 1 \right]$$

$$= \frac{2}{\alpha^2}$$

Now,

$$Var(X) = E(X^2) - (E(X))^2$$

$$= \frac{2}{\alpha^2} - \frac{1}{\alpha^2}$$

$$= \frac{1}{\alpha^2}$$ As Required.

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P3) Derive M.g.f of Exponential Distribution. Find Mean & Var using m.g.f

Solution:- By defination,

$$M_X(t) = E(e^{tX})$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \alpha \int_0^{\infty} e^{tx} e^{-\alpha x}\,dx$$

$$= \alpha \int_0^{\infty} e^{-(\alpha - t)x}\,dx$$

$$= \alpha \left| \frac{e^{-(\alpha-t)x}}{-(\alpha-t)} \right|_0^{\infty}$$

$$= \frac{\alpha}{t-\alpha} \left\{ e^{-\infty} - e^{0} \right\}$$

$$M_X(t) = \frac{-\alpha}{t-\alpha}$$

$$\Rightarrow \quad M_X(t) = \frac{\alpha}{\alpha - t}$$

which is m.g.f

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$$\mu_1' = \frac{d}{dt} M_x(t) \Big|_{t=0}$$

$$= \frac{-\alpha}{(\alpha-t)^2}(-1) \Big|_{t=0}$$

$$= \frac{\alpha}{\alpha^2}$$

$$\text{Mean} = \mu_1' = \frac{1}{\alpha}$$

$$\mu_2' = \frac{d^2}{dt^2} M_x(t) \Big|_{t=0}$$

$$= \frac{-2\alpha}{(\alpha-t)^3}(-1) \Big|_{t=0}$$

$$\mu_2' = \frac{2\alpha}{\alpha^3} \Rightarrow \frac{2}{\alpha^2}$$

$$\Rightarrow Var(X) = \mu_2' - (\mu_1')^2$$

$$= \frac{2}{\alpha^2} - \left(\frac{1}{\alpha}\right)^2$$

$$= \frac{2}{\alpha^2} - \frac{1}{\alpha^2}$$

$$Var(X) = \frac{1}{\alpha^2}$$

As Required.

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P4) Write a short note on characterstic Function.

CHARACTERSTIC Function:-

The characterstic function of random variable X is denoted by $\varphi_X(t)$ & is defined as "The expected value of the random variable e^{itx}" i.e,

$$\varphi_x(t) = E(e^{itx})$$

$$= \sum e^{itx} f(x) \quad \text{(Discrete Dist)}$$

$$= \int_{-\infty}^{\infty} e^{itx} f(x)\,dx \quad \text{(Continous Dist)}$$

The characterstic function may be written as,

$$\varphi_x(t) = f(x) \sum \left(1 + itx + \frac{(it)^2}{2!}x^2 + \frac{(it)^3}{3!}x^3 + \dots \frac{(it)^k}{k!}x^k + \dots\right)$$

$$\varphi_x(t) = \sum f(x) + it \sum x f(x) + \frac{(it)^2}{2!} \sum x^2 f(x) + \frac{(it)^3}{3!} \sum f(x) x^3 + \dots \frac{(it)^k}{k!} \sum x^k f(x) + \dots$$

$$\varphi_x(t) = 1 + it\mu_1' + \frac{(it)^2}{2!}\mu_2' + \frac{(it)^3}{3!}\mu_3' + \cdots + \frac{(it)^k}{k!}\mu_k' + \cdots$$

Thus the kth moment of X about origin is the co-efficient of $\frac{(it)^k}{k!}$

OR

$$\mu_r' = (-1)^r i^r \left[\frac{d^r}{dt^r} \varphi_x(t) \right]_{t=0}$$

Problem 5) Obtain the characteristic function of an Exponential Distribution, Also calculate 1st four central moments.

Solution:- As we know that

$$\varphi_x(t) = E(e^{itx})$$

$$= \int_0^\infty e^{itx} \alpha e^{-\alpha x} dx$$

$$\varphi_x(t) = \alpha \int_0^\infty e^{(it-\alpha)x} dx = \alpha \int_0^\infty e^{-(\alpha-it)x} dx$$

$$\varphi_x(t) = \alpha \left| \frac{e^{-(-it+\alpha)x}}{-(-it+\alpha)} \right|_0^\infty$$

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$$\varphi_x(t) = \frac{-\alpha}{\alpha - it} \left| \frac{1}{e^{(\alpha - it)x}} \right|_0^{\infty}$$

$$= \frac{-\alpha}{\alpha - it} \left(\frac{1}{\infty} - \frac{1}{1} \right]$$

$$\varphi_x(t) = \frac{\alpha}{\alpha - it}$$

Now,

$$\mu_1' = (-1)^1 i^1 \left[\frac{d}{dt} \varphi_x(t) \right]_{t=0}$$

$$\mu_1' = -i \left(\frac{d}{dt} \left(\frac{\alpha}{\alpha - it} \right) \right) \Big|_{t=0}$$

$$\mu_1' = -i \left[\frac{-\alpha}{(\alpha - it)^2} \cdot (-i) \right]_{t=0}$$

$$\mu_1' = \frac{\alpha}{\alpha^2}$$

$$\mu_1' = \frac{1}{\alpha} = \frac{1!}{\alpha}$$

$$\mu_2' = (-1)^2 i^2 \left[\frac{d^2}{dt^2} \varphi_x(t) \right]_{t=0}$$

$$= i^2 \left[\frac{2\alpha(i)(-i)}{(\alpha - it)^3} \right]_{t=0}$$

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$$\mu_2' = (-1)\left[\frac{2\alpha i^2}{(\alpha - it)^3}\right]_{t=0}$$

$$\mu_2' = \left[\frac{2\alpha}{\alpha^3}\right]$$

$$\mu_2' = \frac{2}{\alpha^2} = \frac{2!}{\alpha^2}$$

$$\mu_3' = (-1)^3 i^3 \left[\frac{d^3}{dt^3}\varphi_x(t)\right]$$

$$= -i^2 \cdot i \left[\frac{(+2\alpha)(-3) i^2 (-i)}{(\alpha - it)^4}\right]_{t=0}$$

$$= \left[\frac{6\alpha}{\alpha^4}\right]$$

$$= \frac{6}{\alpha^3} = \frac{3!}{\alpha^3}$$

$$\mu_4' = (-1)^4 i^4 \left[\frac{d^4}{dt^4} \frac{6\alpha i^3}{(\alpha - it)^4}\right]$$

$$= \left.\frac{6\alpha i^3 (-4)(-i)}{(\alpha - it)^5}\right|_{t=0}$$

$$= \frac{24\alpha}{\alpha^5} \Rightarrow \frac{24}{\alpha^4}$$

$$\mu_4' = \frac{4!}{\alpha^4}$$ As Required.

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Now, Moments about center

$\mu_1 = 0$

$\mu_2 = \mu_2' - (\mu_1')^2$

$= \frac{2}{\alpha^2} - \frac{1}{\alpha^2}$

$\mu_2 = \frac{1}{\alpha^2}$

$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2\mu_1'^3$

$\mu_3 = \frac{6}{\alpha^3} - 3\left(\frac{1}{\alpha}\right)\left(\frac{2}{\alpha^2}\right) + 2\left(\frac{1}{\alpha^3}\right)$

$\mu_3 = \frac{6}{\alpha^3} - \frac{6}{\alpha^3} + \frac{2}{\alpha^3}$

$\mu_3 = 2/\alpha^3$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$\mu_4 = \frac{24}{\alpha^4} - 4\left(\frac{6}{\alpha^3}\right)\left(\frac{1}{\alpha}\right) + 6\left(\frac{2}{\alpha^2}\right)\left(\frac{1}{\alpha^2}\right) - \frac{3}{\alpha^4}$

$\mu_4 = \frac{24}{\alpha^4} - \frac{24}{\alpha^4} + \frac{12}{\alpha^4} - \frac{3}{\alpha^4}$

$\mu_4 = \frac{9}{\alpha^4}$ As Required.

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Alternative Method:-

Now as we have calculated that

$$\varphi_x(t) = \frac{\alpha}{\alpha - it}$$

$$\varphi_x(t) = \frac{1}{1 - it/\alpha}$$

$$\varphi_x(t) = (1 - it/\alpha)^{-1}$$

$$\ln \varphi_x(t) = (-1)\ln(1 - it/\alpha)$$

$$K(t) = \frac{it}{\alpha} + \frac{1}{2}\left(\frac{it}{\alpha}\right)^2 + \frac{1}{3}\left(\frac{it}{\alpha}\right)^3 + \frac{1}{4}\left(\frac{it}{\alpha}\right)^4 + \cdots$$

$$K(t) = \frac{1}{\alpha}(it)^1 + \frac{(it)^2}{2!}\frac{1}{\alpha^2} + \frac{(it)^3}{3!}\frac{2}{\alpha^3} + \frac{(it)^4}{4!}\frac{6}{\alpha^4} + \cdots$$

$$\Rightarrow \quad K_1 = \frac{1}{\alpha}, \quad K_2 = \frac{1}{\alpha^2}, \quad K_3 = \frac{2}{\alpha^3}, \quad K_4 = \frac{6}{\alpha^4}$$

As we know that

$$K_1 = \mu_1'$$

$$K_2 = \mu_2' - \mu_1'^2 = \mu_2$$

$$K_3 = \mu_3' - 3\mu_1'\mu_2' + 2\mu_1'^3 = \mu_3$$

$$K_4 = \mu_4' - 3\mu_2' - 4\mu_3'\mu_1' - 6\mu_1'^4 + 12\mu_1'^2\mu_2'$$

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$K_4 = \mu_4 - 3\mu_2^2$

So,

$\mu_1 = 0$

$\mu_2 = \frac{1}{\alpha^2}$

$\mu_3 = 2/\alpha^3$

$\mu_4 = K_4 + 3\mu_2^2$

$\mu_4 = \frac{6}{\alpha^4} + 3\left(\frac{1}{\alpha^2}\right)^2$

$\mu_4 = \frac{6}{\alpha^4} + \frac{3}{\alpha^4}$

$\mu_4 = \frac{6+3}{\alpha^4}$

$\mu_4 = \frac{9}{\alpha^4}$

As Required.

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Problem:- If $f(x) = e^{-x}$ $\quad 0 \le x < \infty$

i) Find $P(1 < x < 2)$

ii) Find distribution Function.

iii) Mean & Var of $f(x)$.

Solution:-

i) $P(1 < x < 2) = \int_1^2 e^{-x}\,dx$

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$= -\Big|_1^2 e^{-x} \Rightarrow -\left\{\frac{1}{e^2} - \frac{1}{e}\right\}$

$= \frac{1}{e} - \frac{1}{e^2} \Rightarrow \left(\frac{e-1}{e^2}\right)$

$= 0.2325$

ii) $p.d.f = \int_0^x f(x)\,dx$

$= \int_0^x e^{-x}\,dx$

$= -\Big|_0^x e^{-x} \Rightarrow -\left(\frac{1}{e^x} - 1\right)$

$= 1 - \frac{1}{e^x}$

$p.d.f = \left(\frac{e^x - 1}{e^x}\right)$

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iii) Mean $= E(X) = \int_{-\infty}^{\infty} x\, f(x)$

$= \int_{0}^{\infty} x\, e^{-x}\, dx$

$= \Big|_{0}^{\infty} x \dfrac{e^{-x}}{-1} + \int_{0}^{\infty} e^{-x}\, dx$

$= \Big|_{0}^{\infty} \dfrac{e^{-x}}{-1}$

$= -(0-1)$

Mean $= 1$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)$

$= \int_{0}^{\infty} x^2 \cdot e^{-x}\, dx$

$= \Big|_{0}^{\infty} -x^2 e^{-x} + 2\int_{0}^{\infty} x e^{-x}\, dx$

$= 0 + 2(1)$

$E(X^2) = 2$

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Now,

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= 2 - 1$$

$$= 1 \quad \text{As Required.}$$

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Q) Find the characterstic function of the Bernoulli's Distribution.

Solution:-

$$\phi_x(t) = E(e^{itx})$$

$$= \sum e^{itx} f(x)$$

$$= \sum_{x=0}^{1} e^{itx} p^x (1-p)^{1-x}$$

$$= e^{it(0)} p^0 (1-p)^{1-0} + e^{it(1)} p^1 (1-p)^1$$

$$= 1 - p + pe^{it}$$

$$= q + pe^{it}$$

$$\phi_x(t) = q + pe^{it} \quad \text{As Required.}$$

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Question) Find characterstic function of the Binomial Distribution.

Solution:-

$$\phi_x(t) = E(e^{itx})$$

$$= \sum_{x=0}^{n} e^{itx} \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^{it})^x q^{n-x}$$

$$= \binom{n}{0} (pe^{it})^0 q^{n-0} + \binom{n}{1} (pe^{it})^1 q^{n-1}$$

$$+ \binom{n}{2} (pe^{it})^2 q^{n-2} + \cdots + \binom{n}{n} (pe^{it})^n q^{n-n}$$

$$= q^n + \binom{n}{1} (pe^{it})^1 q^{n-1} + \binom{n}{2} (pe^{it})^2 q^{n-2}$$

$$+ \cdots + (pe^{it})^n$$

$$= (q + pe^{it})^n$$

$$\Rightarrow \phi_x(t) = (q + pe^{it})^n$$

As Required.

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Question) Find characteristic function of Poisson Distribution.

Solution:- $\phi_x(t) = E(e^{itx})$

$$= \sum e^{itx} f(x)$$

$$= \sum_{x=0}^{\infty} e^{itx} \frac{e^{-m} m^x}{x!}$$

$$= e^{-m} \sum_{x=0}^{\infty} \frac{(me^{it})^x}{x!}$$

$$= e^{-m} \left[1 + me^{it} + \frac{(me^{it})^2}{2!} + \cdots \right]$$

$$= e^{-m} \cdot e^{me^{it}}$$

$$= e^{me^{it} - m}$$

$$= e^{m(e^{it} - 1)}$$

As Required.

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Question) Find characteristic function of the Geometerical Distribution.

Solution:-

$$\phi_x(t) = E(e^{itx})$$

$$= \sum e^{itx} f(x)$$

$$= \sum_{x=1}^{\infty} e^{itx} q^{x-1} p$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} (qe^{it})^x$$

$$= \frac{p}{q} \left[qe^{it} + (qe^{it})^2 + (qe^{it})^3 + \cdots \right]$$

$$= \frac{p}{q} qe^{it} \left[1 + qe^{it} + (qe^{it})^2 + \cdots \right]$$

$$= pe^{it} (1 - qe^{it})^{-1}$$

$$\Rightarrow \phi_x(t) = \frac{pe^{it}}{(1 + qe^{it})}$$

As Required.

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Question) Find characteristic Function of Gamma Distribution.

Solution:- $\phi_x(t) = E(e^{itx})$

$$= \int_0^\infty e^{itx} \frac{1}{\Gamma(m)} e^{-x} x^{m-1} dx$$

$$= \frac{1}{\Gamma(m)} \int_0^\infty e^{itx} e^{-x} x^{m-1} dx$$

$$= \frac{1}{\Gamma(m)} \int_0^\infty e^{-x(1-it)} x^{m-1} dx$$

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Put $u = x(1-it)$

$du = dx(1-it)$

$dx = \frac{du}{1-it}$

As $x = 0 \Rightarrow u = 0$

As $x = \infty \Rightarrow u = \infty$

$$\phi_x(t) = \frac{1}{\Gamma(m)} \int_0^\infty e^{-u} \left(\frac{u}{1-it}\right)^{m-1} \frac{du}{1-it}$$

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$$= \frac{1}{\Gamma(m)} \cdot \frac{1}{(1-it)^m} \int_0^\infty e^{-u} u^{m-1}\, du$$

$$= \frac{1}{\Gamma(m)} \cdot \frac{1}{(1-it)^m} \Gamma(m)$$

$$= \frac{1}{(1-it)^m}$$

$$\Rightarrow \phi_x(t) = (1-it)^{-m}$$

AS Required.

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Question) Find Median of Exponential Distribution.

Solution:- Let M be median

$$\int_{-\infty}^{M} f(x)\, dx = \frac{1}{2}$$

$$\int_0^M \alpha e^{-\alpha x}\,dx = \frac{1}{2}$$

$$-\Big|_0^M e^{-\alpha x} = \frac{1}{2}$$

$$-\left[e^{-\alpha M} - e^{0}\right] = \frac{1}{2}$$

$$1 - \frac{1}{e^{\alpha M}} = \frac{1}{2}$$

$$\frac{1}{e^{\alpha M}} = 1 - \frac{1}{2} = \frac{1}{2}$$

$$e^{\alpha M} = 2$$

$$\alpha M = \ln 2$$

$$M = \frac{\ln 2}{\alpha}$$

$\Rightarrow$

$$\text{Median} = \frac{1}{\alpha}(\ln 2)$$

As Required.

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"Normal Distribution"

A Normal Distribution is defined by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty < x < \infty$$
$$\sigma > 0$$

where μ & σ are Parameters, π & e are constants &

$\mu =$ Mean

$\sigma^2 =$ Variance

The Normal distribution has mean "μ" & Variance σ^2 is usually denoted by $N(\mu, \sigma^2)$

If a r.v. X is normally distributed with Mean "μ" & Variance σ^2 is denoted by

$$X \sim N(\mu, \sigma^2)$$

The Graph of Normal distribution, which is a bell shaped symmeterical curve is called Normal curve.

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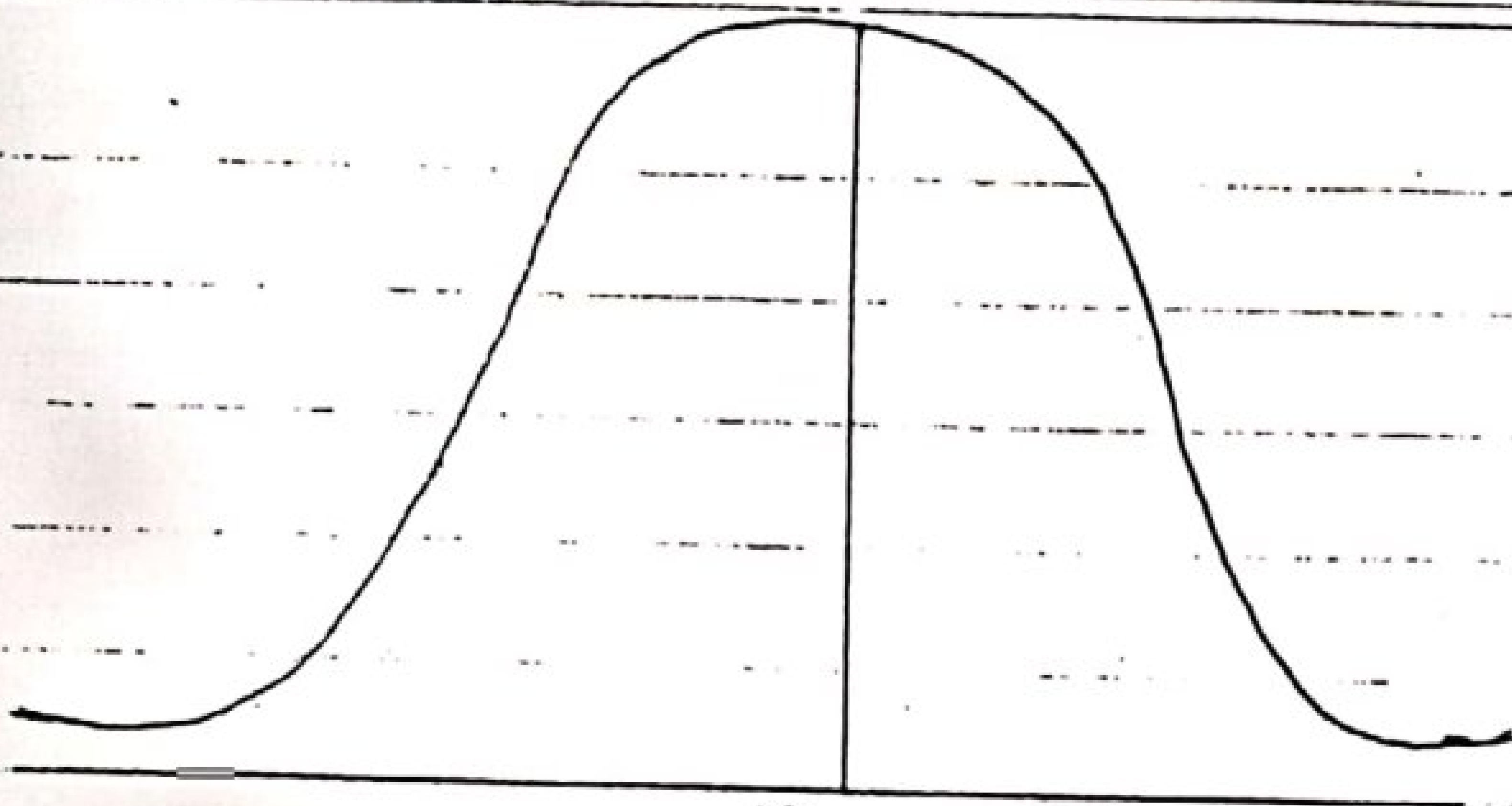

Median, Mean, Mode

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Standard Normal Distribution:-

A Normal Distribution depends on the values of Parameters μ & σ^2 & Various Possible Values for these Parameters will result in unlimited number of different Normal Distribution.

When $\mu = 0$ & $\sigma^2 = 1$ then Normal Dist is called standard Normal Distribution & is denoted as

$X \sim N(0,1)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

for standard Normal Dist

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \qquad -\infty < x < \infty$$

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Question) Show that Area under the Normal curve is Unity.

Solution:-

$$\text{Area} = \int_{-\infty}^{\infty} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

put $\frac{x-\mu}{\sigma} = z$

$$x = \sigma z + \mu$$

$$dx = \sigma\, dz$$

As $x \longrightarrow \infty$ then $z \longrightarrow \infty$

As $x \longrightarrow -\infty$ then $z \longrightarrow -\infty$

So, we have,

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$$\text{Area} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} \sigma\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-z^2/2}\, dz$$

put $z^2/2 = y$

$$2z\,dz = 2\,dy$$

$$z\,dz = dy$$

$$dz = \frac{1}{\sqrt{2y}}\, dy$$

As $z \to \infty$ then $y \to \infty$

As $z \to 0$ then $y \to 0$

~~As $z \to -\infty$ then $z \to \infty$~~

~~$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-y}\, y^{(1/2-1)}\, dy$$~~

$$\Gamma x = \int_0^\infty e^{-t}\, t^{x-1}\, dx$$

$$= \frac{1}{2\sqrt{\pi}}\, 2 \int_0^\infty e^{-y}\, y^{(1/2-1)}\, dy$$

$$= \frac{1}{\sqrt{\pi}}\, \Gamma(1/2) \Rightarrow \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi}$$

$= 1$ As Required.

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Problem) Find Mean, Median & Mode of the Normal Distribution.

Solution:-

Mean)

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$E(X) = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x\, e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$$

put $\frac{x-m}{\sigma} = z$

$x = \sigma z + m$

$dx = \sigma dz$

As $x \longrightarrow \infty$ then $z \longrightarrow \infty$

As $x \longrightarrow -\infty$, then $z \longrightarrow -\infty$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z + m)\, e^{-\frac{1}{2}z^2} \sigma\, dz$$

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$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (m + \sigma z)\, e^{-z^2/2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} m\, e^{-z^2/2}\, dz + \sigma \int_{-\infty}^{\infty} z\, e^{-z^2/2}\, dz \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} m\, e^{-z^2/2}\, dz + (-\sigma) \Big| e^{-z^2/2} \Big|_{-\infty}^{\infty} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} m\, e^{-z^2/2}\, dz + 0 \right]$$

$$= \frac{m}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz$$

Put $z^2/2 = y \Rightarrow z^2 = 2y$

$$2z\,dz = 2\,dy$$

$$z\,dz = dy$$

$$dz = \frac{dy}{\sqrt{2y}}$$

$$= \frac{m}{\sqrt{2}\sqrt{\pi}} \cdot \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-y} \frac{1}{\sqrt{y}}\, dy$$

$$= \frac{m}{2\sqrt{\pi}}\, 2 \int_{0}^{\infty} e^{-y} (y)^{(1/2 - 1)}\, dy$$

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$$= \frac{m}{\sqrt{\pi}} \cdot \Gamma(1/2)$$

$$= \frac{m}{\sqrt{\pi}} \sqrt{\pi}$$

$$= m$$

$\Rightarrow$ **Mean = m**

Median) Assuming median to be a

So,

$$\int_{-\infty}^{a} f(x)\,dx = 1/2$$

$$\int_{-\infty}^{a} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx = 1/2$$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx = 1/2$$

Put $\frac{x-m}{\sigma} = z$

$x = \sigma z + m \Rightarrow dx = \sigma dz$

As $x \rightarrow a$ then $z = a-m/\sigma$

As $x \rightarrow -\infty$ then $z \rightarrow -\infty$

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$$\Rightarrow \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\frac{a-m}{\sigma}} e^{-z^2/2} \sigma \, dz = 1/2$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{a-m}{\sigma}} e^{-z^2/2} dz = 1/2 \rightarrow (A)$$

Also,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-z^2/2} dz = 1/2 \rightarrow (B)$$

By Comparing we get

$$\frac{a-m}{\sigma} = 0$$

$$a - m = 0$$

$$a = m$$

$\Rightarrow$ **Median = m**

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Mode) As

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

$$f'(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-m}{\sigma}\right)^2} 2\left(\frac{-1}{2}\right)\left(\frac{x-m}{\sigma}\right)\left(\frac{1}{\sigma}\right)$$

$$f'(x) = \frac{-1}{\sigma^2\sqrt{2\pi}} e^{-1/2\left(\frac{x-m}{\sigma}\right)^2} \left(\frac{x-m}{\sigma}\right)$$

put $f'(x) = 0$

$$\Rightarrow \quad \frac{x-m}{\sigma} = 0 \quad \because \frac{-1}{\sigma^2\sqrt{2\pi}} e^{-1/2\left(\frac{x-m}{\sigma}\right)^2} \neq 0$$

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$$x - m = 0$$

$$\Rightarrow \quad x = m$$

$$f''(x) = \frac{-1}{\sigma^2\sqrt{2\pi}} \left[e^{-1/2\left(\frac{x-m}{\sigma}\right)^2} 2\left(\frac{-1}{2}\right)\left(\frac{x-m}{\sigma}\right)\left(\frac{1}{\sigma}\right) + e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}\left(\frac{1}{\sigma}\right)\right]$$

put $x = m$

$$f''(x) = \frac{-1}{\sigma^2\sqrt{2\pi}}\left[\frac{1}{\sigma}\right]$$

$$f''(x) = \frac{-1}{\sigma^3\sqrt{2\pi}} < 0$$

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$\Rightarrow$

Mode = m

Hence Normal Distribution is Symmeterical i.e

Mean = Median = Mode = m

Problem) Find m.g.f of Normal Distribution.

solution) :-

$$M_x(t) = E(e^{tx})$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$$

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Put $\frac{x-m}{\sigma} = z$

$$x = m + \sigma z$$

$$dx = \sigma dz$$

As $x \to \pm\infty \Rightarrow z \to \pm\infty$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(m+\sigma z)} e^{-z^2/2} \sigma dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tm} e^{t\sigma z} \cdot e^{-z^2/2} dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{mt} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2t\sigma z)} dz$$

$$= \frac{e^{mt}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2t\sigma z + t^2\sigma^2 - t^2\sigma^2)} dz$$

$$= \frac{e^{mt}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[(z-\sigma t)^2 - t^2\sigma^2]} dz$$

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$$= \frac{e^{mt}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2} e^{t^2\sigma^2/2} dz$$

$$= \frac{e^{mt} \cdot e^{\frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2} dz$$

Put $z - t\sigma = y$

$z = t\sigma + y$

$dz = dy$

As $z \to \pm\infty \Rightarrow y \to \pm\infty$

$$= \frac{e^{mt+\frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy$$

Put $y^2/2 = u$

$y\,dy = du$

$$dy = \frac{du}{\sqrt{2u}}$$

As $y \to \pm\infty \Rightarrow u \to \pm\infty$

So,

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$$= \frac{e^{mt+\frac{\sigma^2 t^2}{2}}}{\sqrt{2}\cdot\sqrt{\pi}} \cdot \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-u} \frac{1}{\sqrt{u}}\, du$$

$$= \frac{e^{mt+\frac{\sigma^2 t^2}{2}}}{2\sqrt{\pi}}\, 2 \int_{0}^{\infty} e^{-u}\, u^{(1/2-1)}\, du$$

$$= \frac{e^{mt+\frac{\sigma^2 t^2}{2}}}{\sqrt{\pi}}\, \Gamma(1/2)$$

$$= \frac{e^{mt+\frac{\sigma^2 t^2}{2}}}{\sqrt{\pi}} \cdot \sqrt{\pi}$$

$$= e^{mt+\frac{\sigma^2 t^2}{2}}$$

$$\Rightarrow M_x(t) = e^{mt+\frac{\sigma^2 t^2}{2}}$$

which is Required m.g.f of Normal Distribution.

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Problem:- Find m.g.f of Normal dist, Find K_1 & K_2.

OR

Prove that in a Normal Distribution all cumulants after 2nd are zero.

Solution:- As we know,

$$M_x(t) = e^{mt + \frac{1}{2}\sigma^2 t^2}$$

$$\ln(M_x(t)) = mt + \frac{1}{2}\sigma^2 t^2$$

$$K(t) = m\frac{t}{1!} + \sigma^2 \frac{t^2}{2!}$$

$$K_1 = m, \; K_2 = \sigma^2, \; K_3 = K_4 = K_5 = \cdots = 0$$

Hence all the cumulants after 2nd are zero.

As Required.

Notes:- In the Examination You must prove 1st of all its M.g.f & then proceeds as above.

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Problem :- Find Mean & Var of Normal Dist whose m.g.f is $e^{166t+200t^2}$

Solution:- Given

$$M_x(t) = e^{166t+200t^2} \longrightarrow (1)$$

& For Normal Distribution

$$M_x(t) = e^{mt+\frac{1}{2}\sigma^2 t^2} \longrightarrow (2)$$

Comparing Eq (1) & (2) we get

$$\Rightarrow m = 166$$

$$\frac{\sigma^2}{2} = 200$$

$$\Rightarrow \sigma^2 = 400$$

As Required.

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Problem :- If p.d.f of a Normal Variable X is

$$f(x) = \frac{1}{\sqrt{32\pi}} e^{-\frac{(x+7)^2}{32}} \qquad -\infty < x < \infty$$

Find its
Mean
Var & m.g.f.

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Solution:-

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-m)^2}{\sigma^2}} \rightarrow (1)$$

$$f(x) = \frac{1}{4\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-(-7))^2}{16}} \rightarrow (2)$$

By comparing Eq (1) & (2)

$m = -7, \quad \sigma = 4 \Rightarrow \sigma^2 = 16$

Now,

$$M_X(t) = e^{mt + \frac{1}{2}\sigma^2 t^2}$$

$$= e^{-7t + \frac{1}{2}16(t^2)}$$

$$= e^{-7t + 8t^2}$$

As Required.

1992:

Problem:- If

$X \sim N(1, 4)$

$Y \sim N(2, 3)$

Find the distribution of $X + 2Y$.

Solution:- As

$X \sim N(1, 4)$

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$\Rightarrow \quad E(X) = 1, \quad Var(X) = 4$

Also, $\quad Y \sim N(2, 3)$

$\Rightarrow \quad E(Y) = 2, \quad Var(Y) = 3$

let $\quad Z = X + 2Y$

$E(Z) = E[X + 2Y]$

$E(Z) = E(X) + 2E(Y)$

$= 1 + 2(2)$

$= 1 + 4 = 5$

$Var(Z) = Var[X + 2Y]$

$= Var(X) + Var(2Y)$

$= Var(X) + 2^2 Var(Y) \quad \because Var(cX) = c^2 Var(X)$

$= 4 + 4(3)$

$= 16$

$\Rightarrow (X + 2Y) \sim (5, 16)$

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Also,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} \quad -\infty < x < \infty$$

$$\Rightarrow \quad f(x) = \frac{1}{16\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-4)^2}{16}} \quad -\infty < x < \infty$$

As Required.

Problem :- $F(x) = K e^{-\frac{(x^2-6x+9)}{24}}$

is the Eq of the Normal curve, Find the value of K, then mean & standard Deviation.

Solution :- As, $F(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-m)^2}{\sigma^2}} \rightarrow (1)$

Given

$$f(x) = K e^{-\frac{1}{2}\frac{(x-3)^2}{12}} \rightarrow (2)$$

By Comparing (1) & (2)

$$m = 3, \quad \sigma^2 = 12 \Rightarrow \sigma = \sqrt{12}$$

$$K = \frac{1}{\sigma\sqrt{2\pi}} = \frac{1}{\sqrt{12}\sqrt{2\pi}}$$

$$\Rightarrow K = 0.1151$$ As Required.

Problem) $X \sim N(0,1)$

find dist of $2X-3$, $\frac{7}{8}X+5$ & $4X$.

Solution) Given

$E(X) = 0$,

$Var(X) = 1$

let $Z = 2X-3$

$E(Z) = E[2X-3]$

$= 2E(X) - E(3)$

$= 2(0) - 3 = -3$

$Var(Z) = Var[2X-3]$

$= 2^2 Var(X) - Var(3)$

$= (2)^2(1) - 0$

$= (2)^2 = 4$

$\because Var(c) = 0$

$\Rightarrow (2X-3) \sim (-3, 2)$

As Required.

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let $Z = \frac{7}{8}x + 5$

$E(Z) = \frac{7}{8}E(x) + 5$

$= \frac{7}{8}(0) + 5 = 5$

$Var(Z) = Var\left(\frac{7}{8}x + 5\right)$

$= \frac{7^2}{8^2}Var(x) + Var(5)$

$= \frac{7}{8}(1) + 0 \Rightarrow \frac{7^2}{8^2}$

$\Rightarrow \left(\frac{7}{8}x + 5\right) \sim \left(5, (7/8)^2\right)$

As Required.

let $Z = 4x$

$E(Z) = 4\,E(x)$

$= 4(0) = 0$

$Var(Z) = 4^2\,Var(x)$

$= 16(1) = 16$

$\Rightarrow (4x) \sim (0, 16)$

As Required.

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* 1999.

Problem:- If x has a Normal Distribution with Mean μ & S.D "σ" then Show that

$$Z = \frac{x - \mu}{\sigma}$$

has standard Normal Distribution.

Solution:- Given that

$$E(X) = \mu, \quad Var(X) = \sigma^2$$

$$S.D(X) = \sigma$$

$$Z = \frac{x - \mu}{\sigma}$$

Z will be a standard Normal Dist. if

i) $E(Z) = 0$

ii) $Var(Z) = 1$

Now,

$$E(Z) = \frac{1}{\sigma}\left[E(X) - E(\mu)\right]$$

$$= \frac{1}{\sigma}\left[\mu - \mu\right]$$

$$= 0$$

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$$Var(Z) = Var\left[\frac{X-\mu}{\sigma}\right]$$

$$= \frac{1}{\sigma^2}\left[Var(X) - Var(\mu)\right]$$

$$= \frac{1}{\sigma^2}\left[\sigma^2 - 0\right]$$

$$= 1$$

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Hence Z is standard Normal Variable.

Problem:- Show that points of inflexion of Normal Dist are Equidistance from Mean.

OR

Find the Points of inflexion of the Normal curve & show that they occure at a distance σ from Mean.

Solution:- By a Point of inflexion we mean a point where curve changes its concavity. Points of inflexion are found by

Solving $f''(x) = 0$

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As we know that

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} \quad -\infty < x < \infty$$

$$f'(x) = \frac{1}{\sigma\sqrt{2\pi}} \left[e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} - \left(\frac{x-m}{\sigma}\right)\left(\frac{1}{\sigma}\right) \right]$$

$$f'(x) = \frac{-1}{\sigma^3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} (x-m)$$

$$f''(x) = \frac{-1}{\sigma^3\sqrt{2\pi}} \left[e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} (-)\left(\frac{x-m}{\sigma}\right)^2\left(\frac{1}{\sigma}\right) + e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} \right]$$

$$= \frac{+1}{\sigma^5\sqrt{2\pi}} \left[e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} (x-m)^2 \right]$$

$$- \frac{1}{\sigma^3\sqrt{2\pi}} \left[e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} \right]$$

Put $f''(x) = 0$

$$\frac{1}{\sigma^3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} \left[\frac{1}{\sigma^2}(x-m)^2 - 1 \right] = 0$$

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As $\frac{1}{\sigma^3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} \neq 0$

$\Rightarrow \quad \frac{(x-m)^2}{\sigma^2} - 1 = 0$

$\frac{(x-m)^2}{\sigma^2} = 1$

$\Rightarrow \quad x-m = \pm\sigma$

$\Rightarrow \quad x = m \pm \sigma$

at $x = m+\sigma$

$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$

$y = f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}} = \frac{1}{\sigma\sqrt{e}\sqrt{2\pi}}$

$y = \frac{1}{\sigma\sqrt{2\pi e}}$

At $x = m - \sigma$

$y = f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2}$

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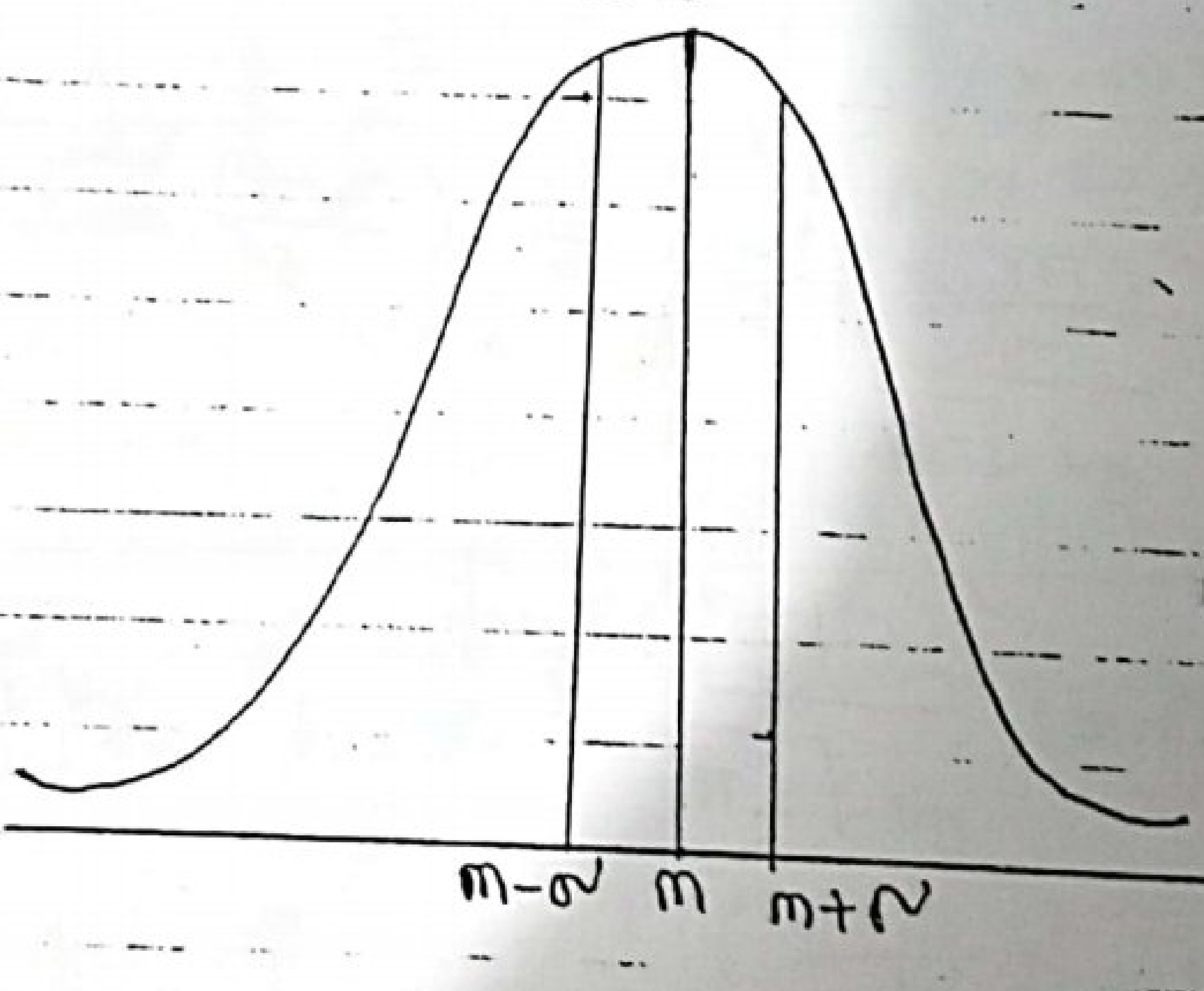
$$\Rightarrow \quad y = f(x) = \frac{1}{\sigma\sqrt{2\pi e}}$$

$$\Rightarrow \quad A\left(m+\sigma, \frac{1}{\sigma\sqrt{2\pi e}}\right)$$

$$B\left(m-\sigma, \frac{1}{\sigma\sqrt{2\pi e}}\right)$$

Hence Point of inflexion are Equidistance from Mean.



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As Required.

2001.*

Problem:- Find m.g.f of Normal Dist about mean M, also find β_1 & β_2.

Solution:- $M_{\bar{x}}(t) = E\left[e^{t(x-\bar{x})}\right]$

$= E\left[e^{tx} \cdot e^{-t\bar{x}}\right]$

$= E\left[e^{tx} \cdot e^{-tm}\right] \quad \because \bar{x} = m$

$= e^{-mt} E(e^{tx})$

$= e^{-mt} M_x(t)$

$M_{\bar{x}}(t) = e^{-mt}\left(e^{mt + \frac{1}{2}\sigma^2 t^2}\right)$

$M_{\bar{x}}(t) = e^{\frac{\sigma^2 t^2}{2}}$

$M_{\bar{x}}(t) = 1 + \left(\frac{1}{2}\sigma^2 t^2\right) + \frac{\left(\frac{1}{2}\sigma^2 t^2\right)^2}{2!} + \frac{\left(\frac{1}{2}\sigma^2 t^2\right)^3}{3!}$

$+ \frac{\left(\frac{1}{2}\sigma^2 t^2\right)^4}{4!} + \cdots$

$= 1 + \frac{\sigma^2 t^2}{2!} + \frac{\sigma^4 t^4}{2^2\, 2!} + \frac{\sigma^3 t^6}{2^3\, 3!} + \cdots$

$= 1 + \sigma^2\left(\frac{t^2}{2!}\right) + 3 \cdot \sigma^4 \frac{t^4}{4!} + \cdots$

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By comparing coefficient of $\frac{t^r}{r!}$ we get

$$\mu_1 = \mu_3 = \mu_5 = \cdots = 0$$

$$\mu_2 = \sigma^2, \quad \mu_4 = 3\sigma^4.$$

Now,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}$$

$$= 0$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

$$= \frac{3\sigma^4}{(\sigma^2)^2}$$

$$= \frac{3\sigma^4}{\sigma^4} = 3$$

As Required.

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Problem) For a Normal Dist $N(m, \sigma^2)$ show that

$$\mu_{2n+2} = \sigma^2 \mu_{2n} + \sigma^3 \frac{d}{d\sigma} \mu_{2n}$$

Also calculate β_2.

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OR

Find Recurrance Relation for Normal Distribution.

Solution:- $\mu_{2n} = E(x - \bar{x})^{2n}$

$$= \int_{-\infty}^{\infty} (x - \bar{x})^{2n} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} (x - \bar{x})^{2n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$$

$$= \int_{-\infty}^{\infty} \frac{(x-m)^{2n}}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} \cdot e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$$

$$= \int_{-\infty}^{\infty} \frac{(x-m)^{2n}}{\sqrt{2\pi}} \frac{1}{\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$$

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$$\frac{d}{d\sigma}\mu_{2n} = \frac{(x-m)^{2n}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{-1}{\sigma^2}e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} + \frac{1}{\sigma}e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} 2\left(\frac{-1}{2}\right)\left(\frac{x-m}{\sigma}\right)(x-m)\cdot\frac{-1}{\sigma^2}\right)$$

$$\frac{d}{d\sigma}\mu_{2n} + \frac{1}{\sigma^2\sqrt{2\pi}}(x-m)^{2n}\int_{-\infty}^{\infty}\frac{-1}{\sigma^2}e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}dx$$

$$= \frac{1}{\sigma^4\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-m)^{2n+2}e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}dx$$

Multiplying by σ^3

$$\sigma^3\frac{d}{d\sigma}\mu_{2n} + \frac{\sigma^2}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-m)^{2n}e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-m)^{2n+2}e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}dx$$

$$\sigma^3\frac{d}{d\sigma}\mu_{2n} + \sigma^2\mu_{2n} = \mu_{2n+2}$$

$$\Rightarrow \mu_{2n+2} = \sigma^2\mu_{2n} + \sigma^3\frac{d}{d\sigma}\mu_{2n}$$

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which is Required Recurrance Relation for Normal Dist.

Put $r = 0$

$$\mu_2 = \sigma^2(1) + \sigma^3 \frac{d}{d\sigma}(1)$$

$$\mu_2 = \sigma^2$$

Put $r = 1$

$$\mu_4 = \sigma^2 \mu_2 + \sigma^3 \cdot \frac{d}{d\sigma}\mu_2$$

$$= \sigma^2(\sigma^2) + \sigma^3(2\sigma)$$

$$= \sigma^4 + 2\sigma^4$$

$$= 3\sigma^4$$

Now

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

$$= \frac{3\sigma^4}{\sigma^4}$$

$$= 3$$ As Required.

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Problems:- NORMAL Approximation to the Binomial Distribution.

OR

Prove that Normal distribution is limiting form of Binomial Dist as $n \longrightarrow \infty$.

Solution:- The P.d.f of Binomial is given by

$$f(x) = \binom{n}{x} p^{x} q^{n-x} \qquad x = 0,1,2,\cdots,n$$

$$\mu = E(X) = np$$

$$\sigma^2 = Var(X) = npq$$

$$S.D(X) = \sqrt{npq}$$

To derive limiting form, we define a new random Variable Z by relation

$$Z = \frac{x - np}{\sqrt{npq}}$$

obviously

$$E(Z) = 0$$

$$\& \quad Var(Z) = 1$$

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Thus we have to show that Z is normally distributed on the limit as $n \to \infty$ & P is finite. We shall use m.g.f to prove the theorem,

$$M_Z(t) = E(e^{tZ})$$

$$= E\left[e^{t\left(\frac{x-np}{\sqrt{npq}}\right)}\right]$$

$$= E\left[e^{\frac{tx}{\sqrt{npq}}} \cdot e^{-npt/\sqrt{npq}}\right]$$

$$= e^{-npt/\sqrt{npq}} \; E\left[e^{\frac{tx}{\sqrt{npq}}}\right]$$

$$= e^{\frac{-npt}{\sqrt{npq}}} \sum_{x=0}^{n} e^{tx/\sqrt{npq}} \binom{n}{x} p^x q^{n-x}$$

$$= e^{\frac{-npt}{\sqrt{npq}}} \left[\sum_{x=0}^{n} \binom{n}{x} q^{n-x} \left(p\, e^{\frac{t}{\sqrt{npq}}}\right)^x\right]$$

$$= e^{\frac{-npt}{\sqrt{npq}}} \left[\binom{n}{0} q^{n-0} \left(P e^{t/\sqrt{npq}}\right)^0 + \binom{n}{1} q^{n-1} \left(P e^{t/\sqrt{npq}}\right)^1 + \binom{n}{2} q^{n-2} \left(P e^{t/\sqrt{npq}}\right)^2 + \dots + \binom{n}{n} q^{n-n} \left(P e^{t/\sqrt{npq}}\right)^n\right]$$

$$= e^{-npt/\sqrt{npq}} \left(q + P e^{t/\sqrt{npq}}\right)^n$$

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$$= \left[q\, e^{\frac{-pt}{\sqrt{npq}}} + p\, e^{\frac{t}{\sqrt{npq}} - \frac{pt}{\sqrt{npq}}} \right]^n$$

$$= \left[q e^{\frac{-pt}{\sqrt{npq}}} + p\, e^{\frac{qt}{\sqrt{npq}}} \right]^n$$

$$= \Big[q \Big(1 - \frac{pt}{\sqrt{npq}} + \frac{1}{2!}\left(\frac{pt}{\sqrt{npq}}\right)^2 - \frac{1}{3!}\left(\frac{pt}{\sqrt{npq}}\right)^3 + \frac{1}{4!}\left(\frac{pt}{\sqrt{npq}}\right)^4 - \cdots \Big)$$

$$+ p \Big(1 + \frac{qt}{\sqrt{npq}} + \frac{1}{2!}\left(\frac{qt}{\sqrt{npq}}\right)^2 + \frac{1}{3!}\left(\frac{qt}{\sqrt{npq}}\right)^3 + \frac{1}{4!}\left(\frac{qt}{\sqrt{npq}}\right)^4 + \cdots \Big) \Big]^n$$

$$= \Big[q - \frac{qpt}{\sqrt{npq}} + \frac{1}{2!}\frac{qp^2t^2}{npq} - \frac{1}{3!}\frac{qp^3t^3}{(npq)^{3/2}} + \frac{1}{4!}\frac{qp^4n^4}{(npq)^2} + \cdots$$

$$+ p + \frac{pqt}{\sqrt{npq}} + \frac{1}{2!}\frac{pq^2t^2}{npq} + \frac{1}{3!}\frac{pq^3t^3}{(npq)^{3/2}} + \frac{1}{4!}\frac{pq^4n^4}{(npq)^2} + \cdots \Big]^n$$

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$$M_Z(t) = \left[q+p+\frac{t^2}{2!\,n}(p+q)+\frac{1}{3!}\frac{qpt^3}{(npq)^{3/2}}(q^2-p^2) + \frac{1}{4!}\frac{pqt^4}{(npq)^2}(q^3+p^3)+\ldots\ldots \right]^n$$

$$M_Z(t) = \left[1+\frac{t^2}{2n}+\frac{1}{3!}\frac{qpt^3}{(npq)^{3/2}}(q-p) + \frac{1}{4!}\frac{pqt^4}{(pqn)^2}(q^3+p^3)+\ldots\ldots \right]^n$$

Taking log on both sides

$$\ln M_Z(t) = n\ln\left[1+\frac{t^2}{2n}+\frac{1}{3!}\frac{pqt^3}{(npq)^{3/2}}(q-p) + \text{terms involving higher powers of } n \text{ in denominator}\right]$$

$$\ln M_Z(t) = n\left[\frac{t^2}{2!\,n}+\frac{1}{3!}\frac{qpt^3}{(npq)^{3/2}}(q-p) + \text{terms involving higher powers of } n \text{ in denominator}\right]$$

$$= \frac{t^2}{2!}+\frac{1}{3!}\frac{(npq)t^3}{(npq)^{3/2}}(q-p) +$$

terms involving higher powers of n in denominator.

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$$\lim_{n \to \infty} \ln M_Z(t) = \frac{t^2}{2!}$$

$$\ln \lim_{n \to \infty} M_Z(t) = t^2/2$$

$$\lim_{n \to \infty} M_Z(t) = e^{t^2/2}$$

Which is m.g.f of Standard Normal Variable.

1991 *

Problem:- If X is Normally distributed with Mean m & S.D σ then $y = \frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2$

is γ-Variate with Parameter $\frac{1}{2}$.

Solution:- As $X \sim N(m, \sigma^2)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

Now,

$$1 = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$$

Put $y = \frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2$

$$y = \frac{(x-m)^2}{2\sigma^2}$$

$$(x-m) = \sigma\sqrt{2}\sqrt{y}$$

$$dx = \frac{\sigma}{\sqrt{2y}}\,dy$$

As $x \longrightarrow \pm\infty \Rightarrow y \longrightarrow \pm\infty$

so,

$$I = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-y}\, \frac{\sigma}{\sqrt{2}\cdot\sqrt{y}}\, dy$$

$$I = \frac{1}{\cancel{2}\sqrt{\pi}}\, \cancel{2} \int_{0}^{\infty} e^{-y}\, y^{(1/2-1)}\, dy$$

$$I = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-y}\, y^{(1/2-1)}\, dy$$

$$I = \frac{1}{\Gamma 1/2} \int_{0}^{\infty} e^{-y}\, y^{(1/2-1)}\, dy$$

which is γ distribution with parameter $m = 1/2$

As Required.

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Problem :- NORMAL Approximation to the Poisson Distribution.

OR

Prove that Normal Dist is limiting form of Poisson Distribution as $m \to \infty$.

Solution :- The P.d.f of Poisson is given by,

$$f(x) = \frac{m^x e^{-m}}{x!} \quad ; x = 0, 1, 2, \ldots, \infty$$

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$$\mu = E(X) = m$$

$$\sigma^2 = Var(X) = m$$

$$S.D.\ X = \sqrt{m}$$

To derive the limiting form we define a new random variable Z by relation

$$Z = \frac{x - m}{\sqrt{m}}$$

obviously $E(Z) = 0$

$$Var(Z) = 1$$

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Thus we have to show that Z is normally distributed on the limit as $n \to \infty$

We shall use mgf to prove the theorem.

$$M_Z(t) = E(e^{tZ})$$

$$= E\left[e^{t\left(\frac{x-m}{\sqrt{m}}\right)}\right]$$

$$= E\left[e^{tx/\sqrt{m}} \cdot e^{-t\sqrt{m}}\right]$$

$$= e^{-\sqrt{m}t} E\left[e^{tx/\sqrt{m}}\right]$$

$$= e^{-\sqrt{m}t} \sum e^{tx/\sqrt{m}} \frac{m^x e^{-m}}{x!}$$

$$= e^{-\sqrt{m}t} \cdot e^{-m} \sum \frac{(me^{t/\sqrt{m}})^x}{x!}$$

$$= e^{-(m+\sqrt{m}t)}\left[\frac{(me^{t/\sqrt{m}})^0}{0!} + \frac{(me^{t/\sqrt{m}})^1}{1!} + \frac{(me^{t/\sqrt{m}})^2}{2!} + \frac{(me^{t/\sqrt{m}})^3}{3!} + \cdots\right]$$

$$= e^{-(m+\sqrt{m}t)}\left[1 + me^{t/\sqrt{m}} + \frac{(me^{t/\sqrt{m}})^2}{2!} + \frac{(me^{t/\sqrt{m}})^3}{3!} + \cdots\right]$$

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$$= e^{-\sqrt{m}t} e^{-m} \left[e^{m e^{\frac{t}{\sqrt{m}}}} \right]$$

$$M_Z(t) = e^{-\sqrt{m}t} e^{m(e^{t/\sqrt{m}}-1)}$$

Taking log on both sides we get

$$\log_e M_Z(t) = -\sqrt{m}t + m(e^{t/\sqrt{m}}-1)$$

$$= -\sqrt{m}t + m\left[1 + \frac{t}{\sqrt{m}} + \frac{1}{2!}\left(\frac{t}{\sqrt{m}}\right)^2 + \frac{1}{3!}\left(\frac{t}{\sqrt{m}}\right)^3 + \cdots - 1\right]$$

$$= -\sqrt{m}t + \sqrt{m}t + \frac{1}{2!} m \cdot \frac{t^2}{m} + \frac{1}{3!} m \left(\frac{t}{\sqrt{m}}\right)^3 + \cdots$$

$$\log_e M_Z(t) = \frac{t^2}{2!} + \frac{t^3}{3!\sqrt{m}} + \cdots$$

$$\lim_{m \to \infty} \log_e M_Z(t) = \frac{t^2}{2!}$$

$$\log_e \lim_{m \to \infty} M_Z(t) = \frac{t^2}{2}$$

$$\lim_{m \to \infty} M_Z(t) = e^{t^2/2}$$

which is M.g.f of the Standard Normal Distribution.

As Required.

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Reproductive Property of Normal Variate:-

If X & Y are two independent Normal Variates then X+Y is also a Normal Variate,

$$M_X(t) = e^{m_1 t + \frac{1}{2}\sigma_1^2 t^2}$$

$$M_Y(t) = e^{m_2 t + \frac{1}{2}\sigma_2^2 t^2}$$

$$M_{X+Y}(t) = E\left[e^{t(X+Y)}\right]$$

$$= E\left[e^{tX} \cdot e^{tY}\right]$$

$$= E(e^{tX}) \cdot E(t^{Y})$$

$$= M_X(t) \cdot M_Y(t)$$

$$= e^{m_1 t + \frac{1}{2}\sigma_1^2 t^2} \cdot e^{m_2 t + \frac{1}{2}\sigma_2^2 t^2}$$

$$= e^{(m_1+m_2)t + \frac{1}{2}(\sigma_1^2+\sigma_2^2)t^2}$$

As Required.

Note:- We calculate for $X_1, X_2, X_3, \ldots, X_n$. (Normal Variate)

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96, 97, 99

Problem :- Prove that for Normal distribution (μ, σ^2) the Quartile deviation, the Mean Deviation & Standard deviation are Approximately in the Ratio

$$10 : 12 : 15$$

Solution:-

$$Q \cdot D = \int_{m-Q}^{m+Q} f(x)dx = \frac{1}{2}$$

$$\int_{m-Q}^{m+Q} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx = \frac{1}{2}$$

put $\frac{x-m}{\sigma} = z$

$$\Rightarrow x = \sigma z + m$$

$$dx = \sigma dz$$

As $x = m - Q$ then $z = -Q/\sigma$

As $x = m + Q$ then $z = Q/\sigma$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-Q/\sigma}^{Q/\sigma} e^{-z^2/2} \sigma dz = 1/2$$

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$$\Rightarrow \frac{2}{\sqrt{2\pi}} \int_0^{Q/\sigma} e^{-z^2/2} dz = 1/2$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_0^{Q/\sigma} e^{-z^2/2} dz = 1/4 = 0.25$$

From Area Table we see that Area under the Normal curve from 0 to Z

= 0.25 is at 0.6745

$$\Rightarrow \frac{Q}{\sigma} = \frac{2}{3}$$

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$$\frac{Q}{2} = \frac{\sigma}{3}$$

$$\frac{Q}{10} = \frac{\sigma}{15} \rightarrow (A)$$

The Mean deviation is given by,

$$M.D = E|X - \bar{X}|$$

$$M.D = E|X - m|$$

$$= \int_{-\infty}^{\infty} |x-m| f(x) dx$$

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$$= \int_{-\infty}^{\infty} |x-m| \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$$

put $\frac{x-m}{\sigma} = z$

$$x = m + \sigma z$$

$$dx = \sigma dz$$

As $x \to \pm\infty$ then $z \to \pm\infty$

$$M \cdot D = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} |\sigma z| e^{-z^2/2} \sigma dz$$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |z| e^{-z^2/2} dz \quad \because \sigma > 0$$

$$= \frac{\sigma}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} -z e^{-z^2/2} dz + \int_{0}^{\infty} z e^{-z^2/2} dz \right]$$

$$= \frac{\sigma}{\sqrt{2\pi}} \left[\int_{0}^{\infty} z e^{-z^2/2} dz + \int_{0}^{\infty} z e^{-z^2/2} dz \right]$$

$$= \frac{\sigma}{\sqrt{2\pi}} 2 \int_{0}^{\infty} z e^{-z^2/2} dz$$

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$$= \frac{-2\sigma}{\sqrt{2\pi}} \int_0^\infty e^{-z^2/2}(-z)\,dz$$

$$= \frac{-2\sigma}{\sqrt{2\pi}} \Big| e^{-z^2/2} \Big|_0^\infty$$

$$= \frac{-2\sigma}{\sqrt{2\pi}} \left(e^{-\infty} - e^{0} \right)$$

$$= \frac{2\sigma}{\sqrt{2\pi}}$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}}\,\sigma$$

$$M.D = \sqrt{\frac{2}{\pi}}\,\sigma$$

$$M.D = 0.79795\,\sigma$$

$$M.D = \frac{4}{5}\sigma$$

$$\frac{M.D}{4} = \frac{\sigma}{5}$$

$$\frac{M.D}{12} = \frac{\sigma}{15} \rightarrow (B)$$

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By Combining eq (A) & (B)

$$\frac{Q}{10} = \frac{M.D}{12} = \frac{\sigma}{15}$$

As Required.

Theorem :- If x is β- Variate of 2nd Kind with Parameter l & m i.e, $\beta_2(l,m)$ then its resiprocal is also a β- Variate of same Kind with Parameters interchanged i.e, $\beta_2(m,l)$.

Proof:- Since x is a $\beta_2(l,m)$ Variate So its Probability differential is

$$dP(x) = \frac{x^{l-1}}{\beta(l,m)\,(1+x)^{l+m}}\,dx \qquad 0 \le x < \infty$$

Put $y = 1/x$

$$dx = \frac{-1}{y^2}\,dy$$

As $x = 0 \Rightarrow y = \infty$

As $x = \infty \Rightarrow y = 0$

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$$dP(y) = \frac{(1/y)^{l-1}}{\beta(l,m)\left(1+\frac{1}{y}\right)^{l+m}} \frac{-1}{y^2}\,dy$$

where y ranges from ∞ to 0.

$$dP(y) = \frac{\frac{1}{y^{l-1}}}{\beta(l,m)\,\frac{(y+1)^{l+m}}{y^{l+m}}} \frac{1}{y^2}\,dy \quad 0 \le y < \infty$$

$$dP(y) = \frac{1}{\beta(l,m)} \frac{y^{l+m-l+1-2}}{(y+1)^{l+m}}\,dy$$

$$dP(y) = \frac{1}{\beta(l,m)} \frac{y^{m-1}}{(y+1)^{l+m}}\,dy \quad 0 \le y < \infty$$

which is Probability density function for β_2-Variate.

Hence Resiprocal of β_2-Variate is again β_2-Variate,

As Required.

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Theorem:- To each β Variate of 1st kind there corresponds a pair of β Variates of 2nd kind.

Proof:- let x be a $\beta_1(l,m)$ Variate then

$$dP(x) = \frac{x^{l-1}(1-x)^{m-1}}{\beta(l,m)}\,dx \quad 0 \le x \le 1$$

Put $x = \frac{1}{1+w}$

As $x = 0 \Rightarrow w = \infty$

As $x = \infty \Rightarrow w = 0$

$$dx = \frac{-1}{(1+w)^2}\,dw$$

$$dP(w) = \frac{\left(\frac{1}{1+w}\right)^{l-1}\left(1-\frac{1}{1+w}\right)^{m-1}}{\beta(l,m)} \cdot \frac{-1}{(1+w)^2}\,dw$$

where w ranges ∞ to 0.

$$dP(w) = \frac{1}{\beta(l,m)}\left[\frac{-1}{(1+w)^{l-1}} \cdot \frac{w^{m-1}}{(1+w)^{m-1}} \cdot \frac{1}{(1+w)^2}\,dw\right]$$

$0 \le w < \infty$

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$$dP(W) = \frac{W^{m-1}}{\beta(l,m)\,(W+1)^{l+m}}\,dW$$

$\Rightarrow$ W is a $\beta_2(l,m)$ Variates, As Resiprocal of $\beta_2(l,m)$ is also $\beta_2(l,m)$ Variates.

Hence for each β-Variate of 1st Kind there corresponds a pair of β-Variates of 2nd Kind.

As Required.

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Problem:- Find characteristic Function of Normal Distribution.

Solution:-

$$\phi_x(t) = E(e^{itx})$$

$$= \int_{-\infty}^{\infty} e^{itx} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{itx} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

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$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$$

put $\frac{x-m}{\sigma} = z$

$x = m + \sigma z$

$dx = \sigma dz$

As $x \longrightarrow \pm\infty \Rightarrow z \longrightarrow \pm\infty$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it(m+\sigma z)} e^{-\frac{1}{2}z^2} \sigma dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itm} e^{it\sigma z} e^{-z^2/2} dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{imt} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2it\sigma z)} dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{imt} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2it\sigma z + (it\sigma)^2 - (it\sigma)^2)} dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{imt} \int_{-\infty}^{\infty} e^{-\frac{1}{2}((z - i\sigma t)^2 + t^2\sigma^2)} dz$$

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$$= \frac{e^{imt}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-i\sigma t)^2} e^{-t^2\sigma^2/2} dz$$

$$= \frac{e^{imt} e^{-\frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-i\sigma t)^2} dz$$

Put $z - i\sigma t = y$

$z = y + i\sigma t$

$dz = dy$

As $z \to \pm\infty \Rightarrow y \to \pm\infty$

$$= \frac{e^{imt - \frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dz$$

Put $-y^2/2 = u$

$y dy = du$

$dy = \frac{du}{\sqrt{2u}}$

As $y \to \pm\infty \Rightarrow u \to \pm\infty$

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$$= \frac{e^{imt - t^2\sigma^2/2}}{\sqrt{2\pi}} \cdot \frac{2}{\sqrt{2}} \int_0^\infty e^{-u} u^{(1/2-1)} du$$

$$= e^{imt - t^2\sigma^2/2} \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-u} u^{(1/2-1)} du$$

$$= e^{imt - t^2\sigma^2/2} \frac{1}{\sqrt{\pi}} \Gamma(1/2)$$

$$= e^{imt - t^2\sigma^2/2} \frac{1}{\sqrt{\pi}} \sqrt{\pi}$$

$$\phi_x(t) = e^{imt - t^2\sigma^2/2}$$

Which is Required characteristic function of Normal Distribution.

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Q) Prove that for Binomial distribution $(q+p)^n$

$$K_{r+1} = pq \frac{dK_r}{dp}$$

Proof :- Since we know that

$$K_r = \frac{d^r}{dt^r} \log_e M_x(t) \Big|_{t=0}$$

For Binomial we know,

$$M_x(t) = (q+pe^t)^n$$

$$K_r = \frac{d^r}{dt^r} \log_e (q+pe^t)^n \Big|_{t=0}$$

$$K_r = n \frac{d^r}{dt^r} \log_e (1-p+pe^t) \Big|_{t=0}$$

$$K_r = n \frac{d^r}{dt^r} \log_e (1+p(e^t-1)) \Big|_{t=0} \longrightarrow (1)$$

Now consider

$$\frac{K_r}{dp} = n \frac{d^r}{dt^r} \frac{d}{dp} \log_e (1+p(e^t-1)) \Big|_{t=0}$$

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$$\frac{dK^r}{dp} = n\frac{d^r}{dt^r}\left\{\frac{1}{1+p(e^t-1)}\cdot(e^t-1)\right\}\Bigg|_{t=0}$$

$$pq\frac{K^r}{dp} = n\frac{d^r}{dt^r}\left\{\frac{pq(e^t-1)}{1+p(e^t-1)}\right\}\Bigg|_{t=0} \longrightarrow (A)$$

Now in Eq (1) replace r by $r+1$

$$K_{r+1} = n\frac{d^{r+1}}{dt^{r+1}}\log_e\{1+p(e^t-1)\}\Big|_{t=0}$$

$$= n\frac{d^r}{dt^r}\frac{d}{dt}\log_e\{1+p(e^t-1)\}\Big|_{t=0}$$

$$K_{r+1} = n\frac{d^r}{dt^r}\left\{\frac{1}{1+p(e^t-1)}pe^t\right\}\Bigg|_{t=0}$$

$$K_{r+1} = n\frac{d^r}{dt^r}\left\{\frac{pe^t}{1+p(e^t-1)}\right\}\Bigg|_{t=0} \longrightarrow (B)$$

Now subtracting Eq (A) & (B)

$$K_{r+1} - pq\frac{K_r}{dp} = n\frac{d^r}{dt^r}\left\{\frac{pe^t - pqe^t + pq}{1+p(e^t-1)}\right\}\Bigg|_{t=0}$$

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$$K_{r+1} - pq\frac{K_r}{dp} = n\frac{d^r}{dt^r}\left[\frac{pe^t(1-q)+pq}{1+p(e^t-1)}\right]$$

$$= np\frac{d^r}{dt^r}\left[\frac{pe^t+q}{1+p(e^t-1)}\right]$$

$$= np\frac{d^r}{dt^r}\left[\frac{pe^t+1-p}{1+p(e^t-1)}\right]$$

$$= np\frac{d^r}{dt^r}\left[\frac{1+p(e^t-1)}{1+p(e^t-1)}\right]$$

$$= 0$$

$$K_{r+1} = pq\frac{K_r}{dp}$$

As Required.

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Joint Distribution :-

The distribution of two or more random variables which are observed simultaniously when an experiment is performed is called their Joint Distribution.

A distribution with a single variate is called univariate. Likewise, a distribution involving two, three or many random variables simultaniously is referred as bivariate, trivariate or multivariate.

Bivariate Distribution :-

A table listing all the values of r.v's X & Y togeather with their probabilities $f(x_i, y_j)$ is called joint distribution of two random variables OR bivariate distribution. Thus the joint Probability function also called bivariate Probability function $f(x, y)$ is a function whose value at the point (x_i, y_j) is given by,

$$f(x_i, y_j) = P(X = x_i \text{ and } Y = y_j) \qquad i = 1, 2, \ldots, m \quad j = 1, 2, \ldots, n$$

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A Joint Probability Distribution has the following Properties.

i) $f(x_i, y_j) \geq 0 \quad \forall \; x_i, y_i$

ii) $\sum_i \sum_j f(x_i, y_j) = 1$

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X \ Y	y_1	y_2	------	y_j	------	y_n	$P(X=x_i)$
x_1	$f(x_1, y_1)$	$f(x_1, y_2)$	------	$f(x_1, y_j)$	------	$f(x_i, y_n)$	$g(x_1)$
x_2	$f(x_2, y_1)$	$f(x_2, y_2)$	------	$f(x_2, y_j)$	------	$f(x_2, y_n)$	$g(x_2)$
⋮	⋮	⋮		⋮		⋮	⋮
x_i	$f(x_i, y_1)$	$f(x_i, y_2)$	------	$f(x_i, y_j)$	------	$f(x_i, y_n)$	$g(x_i)$
⋮	⋮	⋮		⋮		⋮	⋮
x_m	$f(x_m, y_1)$	$f(x_m, y_2)$	------	$f(x_m, y_j)$	------	$f(x_m, y_n)$	$g(x_m)$
$P(Y=y_j)$	$h(y_1)$	$h(y_2)$	------	$h(y_j)$	------	$h(y_n)$	1

"MARGINAL PROBABILITY FUNCTION" :-

From the joint Probability function for (X, Y) we can obtain the individual probability functions of X and Y. Such individual probability functions are called marginal Probability functions.

Let $f(x, y)$ be the joint Probability function of two discrete random variables X & Y. Then the marginal Probability function of X is defined as

$$g(x_i) = \sum_{j=1}^{n} f(x_i, y_j)$$

$$= f(x_i, y_1) + f(x_i, y_2) \ldots \text{———} + y(x_i, y_n)$$

$$= P(X = x_i)$$

i.e, the individual Probability function of X is found by adding over the rows of two way table.

Similarly, the marginal Probability function of "Y" is found by adding over the colums as

$$h(y_j) = \sum_{i=1}^{m} P(x_i, y_j) = P(Y = y_j)$$

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Conditional Probability Functions :-

Let X & Y be two discrete r.v's with joint Probability function $f(x,y)$. Then the conditional Probability function for X given Y = y denoted as $f(x \mid y)$ & is defined by

$$f(x_i \mid y_j) = P(X = x_i \mid Y = y_j)$$

$$= \frac{P(X = x_i \ \& \ Y = y_j)}{P(Y = y_j)}$$

$$= \frac{f(x_i, y_j)}{h(y_j)} \quad \begin{array}{l} i = 1, 2, \ldots m \\ j = 1, 2, \ldots n \end{array}$$

where $h(y)$ is marginal Probability & $h(y) > 0$.

Similarly, the conditional Probability function for Y given X = x is denoted as $f(y \mid x)$ & is defined by,

$$f(y_j \mid x_i) = \frac{P(Y = y_j \mid X = x_i)}{P(X = x_i)}$$

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$$= \frac{f(x_i, y_j)}{g(x_i)}$$

where $g(x) > 0$

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Continous Bivariate Distribution :-

The Bivariate Probability density function of Continous r.v's X & Y is an integrable function $f(x,y)$ satisfying the following Properties,

i) $f(x,y) \geq 0 \quad \forall \ (x,y)$

ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

iii) $p(a \leq X \leq b,\ c \leq Y \leq d) = \int_{a}^{b}\int_{c}^{d} f(x,y)\,dx\,dy$

The distribution function of the bivariate r.v (X,Y) is defined by,

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,du\,dv$$

The marginal p.d.f of Continous r.v X is

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

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& the marginal p.d.f of continous r.v's Y is

$$h(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

The conditional p.d.f of the continous r.v X given that Y takes the value "y" is defined to be

$$f(x|y) = \frac{f(x, y)}{h(y)}$$

where $f(x, y)$ & $h(y)$ are respectively the joint p.d.f of X & Y and the marginal p.d.f of Y & $h(y) > 0$

Similarly the conditional p.d.f of the continous r.v Y given that $X = x$ is,

$$p(y|x) = \frac{f(x, y)}{g(x)}$$

provided $g(x) > 0$

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Independence :-

Two discrete r.v's X & Y are said to be statistically independent iff for all possible pairs of values (x_i, y_j), the joint probability function $f(x,y)$ can be expressed as the product of two marginal probability functions, i.e, X & Y are independent if

$$f(x,y) = P(X = x_i \ \& \ Y = y_j)$$

$$= P(X = x_i) . P(Y = y_j) \ \forall \ i \text{ and } j$$

$$= g(x)\, h(y)$$

Similarly two continous r.v's X & Y are said to be independent iff their joint density $f(x,y)$ can be factored in the form

$$f(x,y) = g(x)\, h(y)$$

for all possible values of X & Y.

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Problem 1) let X & Y be two discrete r.v's with the following joint p.d

y \ x	2	4	h(y)
1	0.10	0.15	0.25
3	0.20	0.30	0.50
5	0.10	0.15	0.25
g(x)	0.40	0.60	1.00

Find, $E(X)$, $E(Y)$, $E(X+Y)$, $E(XY)$, $E(2X-3Y)$.

Solution:-

i)

X	g(x)	x g(x)
2	0.40	0.80
4	0.60	2.40

$\sum x\, g(x) = 3.2$

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As we know

$$E(X) = \sum x\, g(x)$$

$$E(X) = 3.2$$

As Required.

ii)

Y	h(y)	y h(y)
1	0.25	0.25
3	0.50	1.50
5	0.25	1.25
		$\Sigma y h(y) = 3.0$

As we know

$E(Y) = \Sigma y h(y) = 3.0$ As Required.

iii) $E(X+Y) = E(X) + E(Y)$

$= 3.2 + 3.0 = 6.2$

iii) $E(XY) = E(X)\,E(Y)$

$= (3.2)(3) = 9.6$

iv) $E(2X-3Y) = E(2X) - E(3Y)$

$= 2.E(X) - 3E(Y)$

$= 2(3.2) - 3(3)$

$= -2.6$ As Required.

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Problem 2) From the following joint p.d of X & Y find Var(X), Var(Y), Cov(X, Y).

X \ Y	0	1	2	3	g(x)
0	0.05	0.05	0.10	0.00	0.20
1	0.05	0.10	0.25	0.10	0.50
2	0.00	0.15	0.10	0.05	0.30
h(y)	0.10	0.30	0.45	0.15	1.00

Solution:-

x	g(x)	x g(x)	x^2 g(x)
0	0.20	0.00	0.00
1	0.50	0.50	0.50
2	0.30	0.60	4.20
		$\Sigma x g(x) = 1.10$	$\Sigma x^2 g(x) = 1.70$

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$E(X) = \sum x\, g(x) = 1.10$

$E(X^2) = \sum x^2\, g(x) = 1.70$

$Var(X) = E(X^2) - (E(X))^2$

$= 1.70 - (1.10)^2$

$= 0.49$

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Y	h(y)	y h(y)	$y^2 h(y)$
0	0.10	0.00	0.00
1	0.30	0.30	0.30
2	0.45	0.90	1.80
3	0.15	0.45	1.35
		$\sum y h(y) = 1.65$	$\sum y^2 h(y) = 3.45$

$E(Y) = \sum y\, h(y) = 1.65$

$E(Y^2) = \sum y^2 h(y) = 3.45$

$Var(Y) = E(Y^2) - (E(Y))^2$

$= 3.45 - (1.65)^2$

$= 0.7275$

$E(XY) = \sum_i \sum_j x_i y_j f(x_i, y_j)$

$= 1 \times 0.10 + 2 \times 0.15 + 2 \times 0.25 + 4 \times 0.10$
$+ 3 \times 0.10 + 6 \times 0.05$

$= 1.90$

$Cov(XY) = E(XY) - E(X)E(Y)$

$= 1.90 - (1.10)(1.65)$

$= 0.085$

As Required.

Problem:- let the bivariate continous r.v. (X, Y) have the joint Probability density given by,

$$f(x,y) = \begin{cases} x^2 + \frac{xy}{3} & 0 \le x \le 1, 0 \le y \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

a) Check that $f(x,y)$ is a p.d.f

b) Find the marginal p.d.f's

c) Find the conditional p.d.f's & verify that $f(x|y)$ is p.d.f

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Solution:- a)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^2\int_0^1 \left(x^2 + \frac{xy}{3}\right) dx\,dy$$

$$= \int_0^2 \left(\frac{x^3}{3} + \frac{x^2 y}{6}\right)\Big|_0^1 dy$$

$$= \int_0^2 \left(\frac{1}{3} + \frac{y}{6}\right) dy$$

$$= \Big|_0^2 \frac{y}{3} + \frac{y^2}{12} \Rightarrow \frac{2}{3} + \frac{4}{12}$$

$$= \frac{8 + 4}{12} = \frac{12}{12} = 1$$

Hence $f(x,y)$ is p.d.f.

b) The marginal p.d.f are

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^2 \left(x^2 + \frac{xy}{3}\right) dy$$

$$= \Big|_0^2 x^2 y + \frac{xy^2}{6}$$

$$= 2x^2 + \frac{2}{3}x$$

$$g(x) = \frac{2}{3}x(1 + 3x) \quad 0 \le x \le 1$$

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$$h(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = \int_0^1 \left(x^2 + \frac{xy}{3}\right) dx$$

$$= \Big|_0^1 \frac{x^3}{3} + \frac{x^2 y}{6}$$

$$= \frac{1}{3} + \frac{y}{6}$$

$$= \frac{1}{6}(2+y) \qquad 0 \le y \le 2$$

c) The Conditional p.d.f of X for given $Y = y$ is,

$$f(x|y) = \frac{f(x,y)}{h(y)}$$

$$= \frac{x^2 + xy/3}{\frac{1}{6}(2+y)}$$

$$= \frac{6x^2 + 2xy}{2+y} \qquad 0 \le x \le 1, \quad 0 \le y \le 2$$

The conditial p.d.f of Y for given $X = x$ is,

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$$f(y|x) = \frac{f(x,y)}{g(x)}$$

$$= \frac{x^2 + xy/3}{\frac{2}{3}x(1+3x)} = \frac{3x+y}{6x+2}$$

$$f(y|x) = \frac{3x+y}{2(1+3x)} \quad 0 \le y \le 2, \quad 0 \le x \le 1$$

To verify that the conditional p.d.f $f(x|y)$ is a p.d.f we have

$$\int_0^1 \frac{6x^2+2xy}{2+y}\, dx = \frac{1}{2+y}\left(2x^3 + x^2 y\right)\Big|_0^1$$

$$= \frac{1}{2+y}(2+y)$$

$$= 1$$

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Hence Proved that conditional p.d.f $f(x|y)$ is a p.d.f.

As Required.

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Problem:- The joint p.d of two discrete r.v's X & Y is given by

$$f(x,y) = \frac{xy^2}{30} \quad \text{for } x = 1,2,3$$
$$y = 1,2$$

Are X & Y are independent.

Solution:- The Marginal p.d for X is,

$$g(x) = \sum_{y=1}^{2} f(x,y)$$

$$= \sum_{y=1}^{2} \frac{xy^2}{30}$$

$$= \sum_{y=1}^{2} \frac{xy^2}{30}$$

$$= \frac{x}{30}(1) + \frac{x}{30}(2)^2$$

$$= x/6 \quad \text{for } x = 1,2,3$$

The marginal p.d for Y is,

$$h(y) = \sum_{x} f(x,y)$$

$$= \sum_{x=1}^{3} \frac{xy^2}{30}$$

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$= \frac{y^2}{30}(1) + \frac{y^2}{30}(2) + \frac{y^2}{30}(3)$

$= \frac{y^2}{5}$ for $y = 1, 2$

clearly, $g(x) \cdot h(y) = \frac{x}{6} \times \frac{y^2}{5}$

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$= \frac{xy^2}{30}$

$= f(x, y)$

$\Rightarrow$ X & Y are independent.

As Required.

Problem) Let X & Y be independent r. Variables with joint p.d.f

$$f(x, y) = \frac{x(1+3y^2)}{4} \qquad 0 < x < 2, \; 0 < y < 1$$

$$= 0 \qquad \text{Elsewhere}$$

Find

$E(X)$

$E(Y)$

$E(X+Y)$ & $E(XY)$.

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Solution:-

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_{0}^{1} \frac{x(1+3y^2)}{4}\, dy$$

$$= \frac{1}{4} \Big|_0^1 xy + xy^3$$

$$= \frac{1}{4} (x+x)$$

$$= \frac{2x}{4} \Rightarrow \frac{x}{2} \quad \text{For } 0<x<2.$$

Now,

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = \int_{0}^{2} \frac{x(1+3y^2)}{4}\, dx$$

$$= \frac{1+3y^2}{4} \Big|_0^2 \frac{x^2}{2}$$

$$= \frac{1+3y^2}{4} (2)$$

$$= \frac{1}{2}(1+3y^2) \quad 0<y<1$$

Now,

$$E(X) = \int_{-\infty}^{\infty} x\, g(x)\, dx = \int_{0}^{2} x\left(\frac{x}{2}\right) dx$$

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$$= \frac{1}{2} \int_0^2 x^2 \, dx$$

$$= \frac{1}{2} \Big|_0^2 \frac{x^3}{3}$$

$$= \frac{1}{2} \left(\frac{8}{3} \right) \Rightarrow \frac{4}{3}$$

$$E(Y) = \int_{-\infty}^{\infty} y \, h(y) \, dy = \int_0^1 y \cdot \frac{1}{2} (1 + 3y^2) \, dy$$

$$= \frac{1}{2} \int_0^1 (y + 3y^3) \, dy$$

$$= \frac{1}{2} \Big|_0^1 \frac{y^2}{2} + \frac{3y^4}{4}$$

$$= 5/8$$

$$E(X+Y) = E(X) + E(Y)$$

$$= 4/3 + 5/8 = 47/24$$

$$E(XY) = E(X) \cdot E(Y) = \frac{4}{3} \times \frac{5}{8}$$

$$= 5/6$$

As Required.

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1997

Problem :- The joint Probability density of X & Y is given by,

$$f(x,y) = K(2x+y) \quad 0<x<1, \; 0<y<2$$

where K is constant, Find

i) The Marginal density of Y

ii) The conditional density of X given Y = y & use it to evaluate $(P \leq 1/2 \mid y = 1)$

Solution:- As we know that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\Rightarrow \int_{0}^{2}\int_{0}^{1} K(2x+y)\,dx\,dy = 1$$

$$K\int_{0}^{2} (x^2+xy)\Big|_0^1\,dy = 1$$

$$K\int_{0}^{2} (1+y)\,dy = 1$$

$$K\Big|_0^2\, y + y^2/2 = 1$$

$$K\,(2+2) = 1$$

$$\Rightarrow \quad K = 1/4$$

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So, $f(x,y) = \frac{1}{4}(2x+y)$; $0<x<1, 0<y<2$

i) The Marginal density of y is

$h(y) = \int_0^1 \frac{1}{4}(2x+y)\,dx$

$= \frac{1}{4} \Big|_0^1 x^2 + xy$

$h(y) = \frac{1}{4}(1+y) \quad 0<y<2$

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ii) The conditional p.d.f of X for given Y = y is,

$f(x|y) = \frac{f(x,y)}{h(y)}$

$= \frac{1/4\,(2x+y)}{1/4\,(1+y)} = \frac{2x+y}{1+y} \quad \begin{matrix} 0<x<1 \\ 0<y<2 \end{matrix}$

Now,

$P(X \leq \frac{1}{2} / y=1) = \int_0^{1/2} \frac{2x+1}{1+1}\,dx = \frac{1}{2}\Big|_0^{1/2} x^2+x$

$= \frac{1}{2}\left(\frac{1}{4}+\frac{1}{2}\right)$

$= 3/8$ As Required.

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Problem:- Given the following joint p.d.f

$$f(x,y) = \frac{1}{8}(6-x-y) \qquad 0 \le x \le 2,\ 2 \le y \le 4$$
$$= 0 \qquad \text{Elsewhere}$$

a) Varify that $f(x,y)$ is a joint p.d.f.

b) Calculate i) $P(X \le 3/2,\ Y \le 5/2)$

ii) $P(X+Y \le 3)$

c) Find $g(x)$ & $h(y)$

d) Find $f(x|y)$ & $f(y|x)$.

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Solution:- a)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \frac{1}{8}\int_0^2\int_2^4 (6-x-y)\,dx\,dy$$

$$= \frac{1}{8}\int_0^2 \left(6y - xy - y^2/2\right)\Big|_2^4 dx$$

$$= \frac{1}{8}\int_0^2 (24-4x-8)-(12-2x-2)\,dx$$

$$= \frac{1}{8}\int_0^2 (6-2x)\,dx$$

$= \frac{1}{8} \Big|_0^2 6x - \frac{2x^2}{2}$

$= \frac{1}{8}(12 - 4)$

$= \frac{8}{8}$

$= 1$

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so, f(x,y) is joint p.d.f.

b) $P(X \leq 3/2, Y \leq 5/2) = \int_0^{3/2} \int_2^{5/2} \frac{1}{8}(6 - x - y)\,dy\,dx$

$= \frac{1}{8} \int_0^{3/2} \left(6y - xy - y^2/2\right) \Big|_2^{5/2} dx$

$= \frac{1}{8} \int_0^{3/2} \left(\left(\frac{30}{2} - \frac{5}{2}x - \frac{25}{8}\right) - (12 - 2x - 2)\right) dx$

$= \frac{1}{8} \int_0^{3/2} \left(\frac{15}{8} - \frac{x}{2}\right) dx$

$= \frac{1}{8} \Big|_0^{3/2} \frac{15}{8}x - x^2$

$= \frac{1}{8}\left(\frac{15}{8} \times 3/2 - \frac{9}{4}\right)$

$= \frac{9}{32}$ <u>As Required.</u>

$$P(X+Y \le 3) = \int_0^1 \int_2^{3-x} \frac{1}{8}(6-x-y)\, dy\, dx \qquad \because x+y \le 3, \; y \le 3-x$$

$$= \frac{1}{8} \int_0^1 \left(6y - xy - \frac{y^2}{2}\right)\Big|_2^{3-x} dx$$

$$= \frac{1}{8} \int_0^1 \Big(6(3-x) - x(3-x) - \frac{1}{2}(9+x^2-6x) - (6(2) - 2x - 2) \Big) dx$$

$$= \frac{1}{8} \int_0^1 \Big(18 - 6x - \cancel{3x} + x^2 + \cancel{3x} - \frac{9}{2} - \frac{x^2}{2} - 12 + 2x + 2 \Big) dx$$

$$= \frac{1}{8} \int_0^1 \Big(\frac{x^2}{2} - 6x + \frac{27}{2} - 12 + 2x + 2 \Big) dx$$

$$= \frac{1}{8} \int_0^1 \Big(\frac{x^2}{2} - 4x + \frac{7}{2}\Big) dx$$

$$= \frac{1}{8} \Big| \frac{x^3}{6} - \frac{4x^2}{2} + \frac{7}{2}x \Big|_0^1$$

$$= \frac{1}{8}\left[\frac{1}{6} - 2 + \frac{7}{2}\right]$$

$$= \frac{1}{8}\left[\frac{1-12+21}{6}\right] = \frac{5}{24}$$

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c) $g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$= \int_{2}^{4} \frac{1}{8}(6-x-y)\,dy$

$= \frac{1}{8} \Big|_{2}^{4} (6y - xy - y^2/2)$

$= \frac{1}{8}\left[(24 - 4x - 8) - (12 - 2x - 2)\right]$

$= \frac{1}{8}\left[6 - 2x\right]$

$= \frac{1}{4}(3-x) \qquad 0 \le x \le 2$

$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

$= \frac{1}{8}\int_{0}^{2} (6-x-y)\,dx$

$= \frac{1}{8} \Big|_{0}^{2} 6x - x^2/2 - xy$

$= \frac{1}{8}\left[12 - 2 - 2y\right]$

$= \frac{1}{4}(5-y) \qquad 2 \le y \le 4$

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d) $$f(x|y) = \frac{f(x,y)}{h(y)}$$

$$= \frac{\frac{1}{8}(6-x-y)}{\frac{1}{4}(5-y)}$$

$$= \frac{6-x-y}{2(5-y)}$$

Now

$$f(y|x) = \frac{f(x,y)}{g(x)}$$

$$= \frac{\frac{1}{8}(6-x-y)}{\frac{1}{4}(3-x)}$$

$$= \frac{6-x-y}{2(3-x)}$$

As Required.

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1997

Problem:- If the Probability density of X is given by,

$$f(x) = Kx^3(1-2x)^6$$

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where K is a constant, find the Probability density function of r.v

$$Y = \frac{2X}{1+2X}$$

Solution:- Given

$$f(x) = K x^3 (1-2x)^6$$

also,

$$Y = \frac{2X}{1+2X}$$

$$\frac{1}{Y} = \frac{1+2X}{2X}$$

$$\frac{1}{Y} = \frac{1}{2X} + 1$$

$$-1 + \frac{1}{Y} = \frac{1}{2X}$$

$$\frac{-Y+1}{Y} = \frac{1}{2X}$$

$$\Rightarrow \frac{1}{X} = \frac{-2(Y-1)}{Y}$$

$$\Rightarrow X = \frac{Y}{-2(Y-1)} = \frac{Y}{2(1-Y)}$$

So,

$$f(y) = K \frac{Y^3}{8(-Y+1)^3} \left[1 - \frac{Y}{-Y+1}\right]^6$$

$$f(y) = \frac{K(Y^3)}{8(1-Y)^3} \left(\frac{-2Y+1}{1-Y}\right)^6$$

As Required.

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Problem:- The Joint Probability Density of X & Y is given by

$$F(x, y) = K(2x + y) \qquad 0 < x < 1$$
$$0 < y < 2$$

where K is constant.

i) Find Marginal density of Y
ii) The conditional density of X given Y = y
iii) & use it to Evaluate

$$P(X \leq 1/2 \mid y = 1)$$

Solution:- As we know

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\int_0^2\int_0^1 K(2x+y)\,dx\,dy = 1$$

$$K\int_0^2 (x^2 + xy)\Big|_0^1\,dy = 1$$

$$K\int_0^2 (1+y)\,dy = 1$$

$$K\Big|_0^2 (y + y^2/2) = 1$$

$$K(2+2) = 1$$

$$\Rightarrow \quad K = 1/4$$

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$\Rightarrow f(x) = \frac{1}{4}(2x+y)$

i) $h(y) = \int_0^1 f(x,y)\,dx$

$= \int_0^1 \frac{1}{4}(2x+y)\,dx$

$= \frac{1}{4} \Big|_0^1 x^2 + xy$

$= \frac{1}{4}(1+y)$

$h(y) = \frac{1+y}{4} \quad 0<y<2$

ii) $f(x/y=y) = \frac{f(x,y)}{h(y)} = \frac{\frac{1}{4}(2x+y)}{\frac{1}{4}(1+y)}$

$= \frac{2x+y}{1+y} \quad \begin{array}{l} 0<x<1 \\ 0<y<2 \end{array}$

iii) $P(X \leq 1/2 \setminus Y=1) = \int_0^{1/2} \frac{2x+1}{2}\,dx$

$= \frac{1}{2} \Big|_0^{1/2} x^2 + x$

$= \frac{1}{2}\left[\frac{1}{4} + \frac{1}{2}\right] = \frac{1}{2}\left[\frac{1+2}{4}\right]$

$= 3/8$ As Required.

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Chi - Square Distribution

Let $z_1, z_2, \ldots, z_n$ be normally & independently distributed variables with zero means & unit variance. Then a random variable expressed by the quantity

$$\chi^2 = \sum_{i=1}^{n} z_i^2$$

is defined as a chi-square random variable with n degree of freedom. That is, a χ^2 random variable is defined as the sum of squares of n independent standard normal variables. Its density function has the following form

$$F(\chi^2) = \frac{1}{2^{n/2}\,\Gamma n/2} (\chi^2)^{n/2 - 1} e^{-\chi^2/2} \qquad 0 < \chi^2 < \infty$$

where n is called Parameter & it is called degree of freedom.

* (1995, 2002)

Problem :-

Derive the chi - square (χ^2) distribution.

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Solution:- To obtain the distribution of $\chi^2 = \sum_{i=1}^{n} z_i^2$ we use the m.g.f technique. The mgf of chi-square random variable is.

$$M_{\chi^2}(t) = E\left[e^{t\chi^2}\right]$$

$$= E\left[e^{t\sum_{i=1}^{n} z_i^2}\right]$$

$$= E\left[e^{t(z_1^2+z_2^2+z_3^2+\cdots\cdots+z_n^2)}\right]$$

$$= E\left[e^{tz_1^2}.e^{tz_2^2}.e^{tz_3^2}\cdots\cdots.e^{tz_n^2}\right]$$

$$= E(e^{tz_1^2}).E(e^{tz_2^2})\cdots\cdots.E(e^{tz_n^2})$$ $\because z_i$ are independent

$$= \prod_{i=1}^{n} E(e^{tz_i^2})$$

Now consider

$$E\left[e^{tz_i^2}\right] = \int_{-\infty}^{\infty} e^{tz_i^2} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{tz_i^2} \frac{1}{\sqrt{2\pi}} e^{-z_i^2/2}\,dz_i$$

$\because z_i$ is normaly distributed.

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$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(1-2t) z_i^2/2} dz_i$$

$$E\left[e^{tz_i^2}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} z_i^2 (1-2t)} dz_i \longrightarrow (1)$$

put $\left(\frac{1-2t}{2}\right)^{1/2} z_i = y_i$

$$z_i = \left(\frac{2}{1-2t}\right)^{1/2} y_i$$

$$dz_i = \left(\frac{2}{1-2t}\right)^{1/2} dy_i$$

So $1 \Rightarrow$

$$E\left[e^{tz_i^2}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y_i^2} \left(\frac{2}{1-2t}\right)^{1/2} dy_i$$

$$= \frac{1}{\sqrt{2\pi}} \frac{\sqrt{2}}{\sqrt{1-2t}} \int_{-\infty}^{\infty} e^{-y_i^2} dy_i$$

$$= \frac{1}{\sqrt{\pi}\sqrt{1-2t}} \int_{-\infty}^{\infty} e^{-y_i^2} dy_i$$

put $y_i^2 = u_i$

$$2y_i \, dy_i = du_i$$

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$$dY_i = \frac{du_i}{2\sqrt{u_i}}$$

AS $Y_i \longrightarrow \pm\infty \Rightarrow u_i \longrightarrow \pm\infty$

So,

$$E(e^{tZ_i^2}) = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{1-2t}} \not{2} \cdot \int_0^\infty \frac{e^{-u_i}(u_i)^{1/2-1}}{\not{2}} du_i$$

$$= \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{1-2t}} \cdot \Gamma\tfrac{1}{2}$$

$$= \frac{\sqrt{\pi}}{\sqrt{\pi} \cdot \sqrt{1-2t}}$$

$$\Rightarrow E(e^{tZ_i^2}) = \frac{1}{\sqrt{1-2t}}$$

So,

$$M_{\chi^2}(t) = \left(\frac{1}{\sqrt{1-2t}}\right)^n$$

$$M_{\chi^2}(t) = \left(\frac{1}{1-2t}\right)^{n/2} \quad \text{for } t < 1/2 \text{ ...... (A)}$$

We know that m.g.f of Gamma distribution

$f(x) = \frac{a^P}{\Gamma P} x^{P-1} e^{-ax}$ is $M_X(t) = \left(\frac{1}{1-\frac{t}{a}}\right)^P$ ...... (B)

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Comparing Eq (A) & (B) we get

$$p = n/2$$

$$a = 1/2$$

Thus p.d.f of χ^2 is

$$f(\chi^2) = \frac{(1/2)^{n/2}}{\Gamma n/2} (\chi^2)^{n/2-1} e^{-\chi^2/2}$$

$$f(\chi^2) = \frac{1}{2^{n/2}\Gamma n/2} (\chi^2)^{n/2-1} e^{-\chi^2/2} \quad 0<\chi^2<\infty$$

which is probability density function of χ^2 distribution.

As Required.

Problem:- Find m.g.f of the chi-square distribution.

Solution:- In case of chi-square distribution

$$f(y) = \frac{1}{2^{n/2}\Gamma n/2} y^{n/2-1} e^{-y/2} \quad 0<y<\infty$$

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The moment Generating function is given by

$$M_Y(t) = E(e^{tY})$$

$$= \int_{-\infty}^{\infty} e^{tY} f(y)\, dy$$

$$= \int_{0}^{\infty} e^{ty} \cdot \frac{1}{2^{n/2}\Gamma n/2}\, y^{n/2-1} e^{-y/2}\, dy$$

$$= \frac{1}{2^{n/2}\Gamma n/2} \int_{0}^{\infty} e^{-y/2(1-2t)}\, y^{n/2-1}\, dy$$

put $\frac{y(1-2t)}{2} = z$

$$y = \frac{2z}{1-2t}$$

$$dy = \left(\frac{2}{1-2t}\right) dz$$

As $y = 0 \Rightarrow z = 0$

$y = \infty \Rightarrow z = \infty$

$$M_Y(t) = \frac{1}{2^{n/2}\Gamma n/2} \int_{0}^{\infty} e^{-z} \left(\frac{2z}{1-2t}\right)^{n/2-1} \left(\frac{2}{1-2t}\right) dz$$

$$= \frac{1}{2^{n/2}\Gamma n/2} \left(\frac{2}{1-2t}\right)^{n/2} \int_{0}^{\infty} e^{-z} z^{n/2-1}\, dz$$

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$$M_Y(t) = \frac{1}{2^{n/2}\,\cancel{\Gamma n/2}}\left(\frac{2}{1-2t}\right)^{n/2} \cancel{\Gamma n/2}$$

$$M_Y(t) = \frac{\cancel{2^{n/2}}}{\cancel{2^{n/2}}\,(1-2t)^{n/2}}$$

$$M_Y(t) = \frac{1}{(1-2t)^{n/2}}$$

$$M_Y(t) = (1-2t)^{-n/2} \quad \text{for } t < 1/2$$

which is Required m.g.f of chi Square Distribution.

Problem:- Find the characterstic function of the chi-square distribution.

Solution:- As we know that

$$\phi_Y(t) = E(e^{itY}) = \int_{-\infty}^{\infty} e^{ity} f(y)\,dy$$

$$= \int_0^{\infty} e^{ity} \frac{1}{2^{n/2}\,\Gamma n/2} y^{n/2-1} e^{-y/2}\,dy$$

$$= \frac{1}{2^{n/2}\,\Gamma n/2} \int_0^{\infty} e^{-y/2(1-2it)} y^{n/2-1}\,dy$$

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Put $\frac{y}{2}(1-2it) = z$

$$\Rightarrow \quad y = \frac{2z}{(1-2it)}$$

$$dy = \left(\frac{2}{1-2it}\right) dz$$

As $y = 0 \Rightarrow z = 0$

& $y = \infty \Rightarrow z = \infty$

$$= \frac{1}{2^{n/2}\,\Gamma n/2} \int_0^\infty e^{-z} \left(\frac{2z}{1-2it}\right)^{n/2-1} \left(\frac{2}{1-2it}\right) dz$$

$$= \frac{1}{2^{n/2}\,\Gamma n/2} \cdot \frac{2^{n/2}}{(1-2it)^{n/2}} \int_0^\infty e^{-z} z^{n/2-1}\, dz$$

$$= \frac{1}{\Gamma n/2} \cdot \frac{1}{(1-2it)^{n/2}} \; \Gamma n/2$$

$$\phi_y(t) = (1-2it)^{-n/2}$$

which is Required ch—stic function of chi-square distribution.

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Problem:- Find Mean & Var of Chi-Square distribution.

Solution:- As we know m.g.f of chi-square distribution is,

$$M_Y(t) = (1-2t)^{-n/2}$$

$$\frac{d}{dt}(M_Y(t)) = \frac{-n}{2}(+1-2t)^{-n/2-1}(-2)$$

$$\frac{d}{dt}(M_Y(t)) = n(1-2t)^{-n/2-1}$$

$$Mean = \Big|_{t=0} \frac{d}{dt}(M_Y(t)) = n(1-0)^{-n/2-1}$$

$$\Rightarrow E(Y) = Mean = \mu_1' = n$$

Now,

$$\frac{d^2}{dt^2}(M_Y(t)) = n\left(\frac{-n}{2}-1\right)(1-2t)^{-n/2-2}(-2)$$

$$= 2n\left(\frac{n}{2}+1\right)(1-2t)^{-n/2-2}$$

$$= n(n+2)(1-2t)^{-n/2-2}$$

$$\mu_2' = \Big|_{t=0} \frac{d^2}{dt^2}(M_Y(t)) = n(n+2)$$

$$\Rightarrow E(Y^2) = \mu_2' = n(n+2)$$

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$Var(Y) = E(Y^2) - [E(Y)]^2$

$= n^2+2n - n^2$

$Var(Y) = 2n$

As Required.

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Problem:- Show that the total Area under the density curve of a chi-square distribution is equal to one.

Solution:- As we know,

$$\therefore f(y) = \frac{1}{2^{n/2}\,\Gamma n/2}\, e^{-y/2}\, y^{n/2-1} \quad 0<y<\infty$$

$$\text{Area} = \int_0^\infty f(y)dy = \int_0^\infty \frac{1}{2^{n/2}\,\Gamma n/2}\, e^{-y/2}\, y^{n/2-1}\, dy$$

$$= \int_0^\infty \frac{1}{2^{n/2}\,\Gamma n/2}\, e^{-y/2}\, y^{n/2-1}\, dy$$

Put $y/2 = z \Rightarrow y = 2z$

$dy = 2\,dz$

As $y = 0 \Rightarrow z = 0$
$y = \infty \Rightarrow z = \infty$

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$$\text{Area} = \frac{1}{2^{n/2}\,\Gamma n/2} \int_0^\infty (2z)^{n/2-1} e^{-z}\, 2\,dz$$

$$= \frac{2^{n/2-1} \cdot 2}{2^{n/2}\,\Gamma n/2} \int_0^\infty e^{-z} z^{n/2-1}\, dz$$

$$= \frac{2^{n/2}}{2^{n/2}} \cdot \frac{1}{\Gamma n/2} \cdot \Gamma n/2$$

Area = 1

Hence Area under the density curve of a chi-square Distribution is equal to one.

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Problem:- Find Cumulant Generating function of χ^2 distribution, Also find β & γ Co-efficients.

Solution:- As we know

$$M_y(t) = (1-2t)^{-n/2}$$

Applying $\log_e$ on both sides,

$$\log_e M_y(t) = \log_e (1-2t)^{-n/2}$$

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$$k(t) = -\frac{n}{2}\log(1-2t)$$

$$k(t) = -\frac{n}{2}\left[-2t - \frac{1}{2}(+2t)^2 - \frac{1}{3}(2t)^3 - \frac{1}{4}(2t)^4 + \cdots\right]$$

$$k(t) = nt + 2n\frac{t^2}{2!} + 8n\frac{t^3}{3!} + 48n\frac{t^4}{4!} + \cdots\cdots$$

$\Rightarrow$ $K_1 = n$

$K_2 = 2n$

$K_3 = 8n$

$K_4 = 48n$

As we know,

$$\mu_1 = 0$$

So,

$$\kappa_2 = \mu_2 = 2n$$

$$\kappa_3 = \mu_3 = 8n$$

$$\mu_4 = K_4 + 3K_2^2 = 48n + 3(2n)^2$$

$$= 48n + 12n^2$$

Now,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(8n)^2}{(2n)^3} = \frac{64n^2}{8n^3} = \frac{8}{n}$$

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$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{48n+12n^2}{(2n)^2}$$

$$= \frac{12n(4+n)}{4n^2}$$

$$= \frac{3(4+n)}{n}$$

Now,

$$\gamma_1 = \sqrt{\beta_1}$$

$$\gamma_1 = \sqrt{8/n}$$

$$\gamma_1 = \frac{2\sqrt{2}}{\sqrt{n}} \Rightarrow 2\sqrt{\frac{2}{n}}$$

$$\gamma_2 = \beta_2 - 3$$

$$\gamma_2 = \frac{3(4+n)}{n} - 3$$

$$\gamma_2 = \frac{12+3n-3n}{n}$$

$$\gamma_2 = 12/n$$

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Problem :- Find Mode & Skeness of chi-square distribution.

Solution :- As we know that

$$f(y) = \frac{1}{2^{n/2}\Gamma n/2} (y)^{n/2 - 1} e^{-y/2} \qquad 0 < y < \infty$$

For mode we put

$$f'(y) = 0 \text{ provided } f''(y) < 0$$

Now,

$$f'(y) = \frac{1}{2^{n/2}\Gamma n/2} \left(\frac{n}{2} - 1\right) y^{n/2 - 2} e^{-y/2}$$

$$+ \frac{1}{2^{n/2}\Gamma n/2} y^{n/2 - 1} e^{-y/2} \left(\frac{-1}{2}\right)$$

$$f'(y) = \frac{1}{2^{n/2}\Gamma n/2} (y)^{n/2 - 1} e^{-y/2} \left[\left(\frac{n}{2} - 1\right)(y)^{-1} - \frac{1}{2}\right]$$

$$f'(y) = f(y) \left[\left(\frac{n}{2} - 1\right) y^{-1} - \frac{1}{2}\right]$$

put $f'(y) = 0$

As $f(y) \neq 0$

So,

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$$\left(\frac{n}{2} - 1\right) y^{-1} - \frac{1}{2} = 0$$

$$\frac{n}{2} - 1 = \frac{1}{2} y$$

$$\frac{n-2}{2} = \frac{1}{2} y$$

$$\boxed{y = n-2}$$

$$f''(y) = f'(y) \left[\frac{n-2}{2} (y)^{-1} - \frac{1}{2} \right] + f(y) \left[-\left(\frac{n-2}{2}\right) y^{-2} \right]$$

$$= f'(y) \left[\frac{n-2}{2y} - \frac{1}{2} \right] - f(y) \left[\frac{n-2}{2y^2} \right]$$

As $y = n-2$

$$f''(y) = 0 - f(n-2) \left[\frac{n-2}{2(n-2)^2} \right]$$

$$f''(y) = -\frac{f(n-2)}{2(n-2)} < 0$$

$\Rightarrow$ $y = n-2$ gives the Mode of χ^2 distribution.

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$$\text{Skewness} = \frac{\text{Mean} - \text{Mode}}{S.D}$$

$$= \frac{n - (n-2)}{\sqrt{2n}}$$

$$= \frac{n - n + 2}{\sqrt{2n}}$$

$$= \sqrt{\frac{2}{n}}$$

As Required.

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Problem:- Prove the additive Property of independent chi-square Variate.

OR

The sum of independent chi-square Variate is also a chi-square Variate.

OR

If χ_i^2 $(i = 1, 2, \ldots, k)$ are independent chi-square variates with n_i d.f respectively, then their sum $\sum_{i=1}^{k} \chi_i^2$ is also a chi-square Variate with $\sum_{i=1}^{k} n_i$ d.f.

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Solution:- We have

$$M_{\chi_i^2}(t) = (1-2t)^{-\frac{n_i}{2}} \quad i = 1, 2, 3, — k$$

Therefor,

$$M_{\Sigma \chi_i^2}(t) = M_{\chi_1^2}(t) \cdot M_{\chi_2^2}(t) \cdot M_{\chi_3^2}(t) \cdots M_{\chi_k^2}(t)$$

$\because \chi_i^2$ being independent.

$$M_{\Sigma \chi_i^2}(t) = (1-2t)^{-n_1/2} \cdot (1-2t)^{-n_2/2} \cdot (1-2t)^{-n_3/2} \cdots (1-2t)^{-n_k/2}$$

$$M_{\Sigma \chi_i^2}(t) = (1-2t)^{-\frac{1}{2}(n_1+n_2+\cdots+n_k)}$$

which is m.g.f of a chi-square variate with $(n_1+n_2+\cdots+n_k)$ degree of freedom.

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Problem :- Prove that As $n \to \infty \Rightarrow \chi^2$-dist $\to$ Normal dist

OR

show that χ^2-dist approaches to Normal dist when $n \to \infty$.

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Solution:- As we know

$$M_{\chi^2}(t) = (1-2t)^{-n/2}$$

putting $Z = \frac{\chi^2 - n}{\sqrt{2n}}$

where n is mean & $\sqrt{2n}$ is the standard deviation of χ^2 distribution.

$$E(Z) = 0$$

$$Var(Z) = 1$$

Now,

$$M_Z(t) = E[e^{tZ}] = E\left[e^{t\left(\frac{\chi^2 - n}{\sqrt{2n}}\right)}\right]$$

$$= E\left[e^{\frac{t\chi^2}{\sqrt{2n}}} \cdot e^{-nt/\sqrt{2n}}\right]$$

$$= e^{-nt/\sqrt{2n}}\left[E\left(e^{t\chi^2/\sqrt{2n}}\right)\right]$$

$$= e^{-nt/\sqrt{2n}} \cdot M_{\chi^2}\left(\frac{t}{\sqrt{2n}}\right)$$

$$= e^{-nt/\sqrt{2n}}\left[1 - \frac{2t}{\sqrt{2n}}\right]^{-n/2}$$

$$\because M_{\chi^2}(t) = (1-2t)^{-n/2}$$

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$$M_z(t) = e^{\frac{-nt}{\sqrt{2n}}}\left[1 - \frac{2t}{\sqrt{2n}}\right]^{-n/2}$$

Applying $\log_e$ on both sides

$$\log_e(M_z(t)) = \frac{-nt}{\sqrt{2n}} - \frac{n}{2}\log\left[1 - \frac{2t}{\sqrt{2n}}\right]$$

we know that

$$\log(1-z) = z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} \cdots$$

So,

$$\log_e M_z(t) = \frac{-nt}{\sqrt{2n}} - \frac{n}{2}\left[-\left(\frac{2t}{\sqrt{2n}}\right) - \frac{1}{2}\left(\frac{2t}{\sqrt{2n}}\right)^2 - \frac{1}{3}\left(\frac{2t}{\sqrt{2n}}\right)^3 - \frac{1}{4}\left(\frac{2t}{\sqrt{2n}}\right)^4 - \cdots\right]$$

$$= \frac{-nt}{\sqrt{2n}} + \frac{n}{2}\left[\frac{2t}{\sqrt{2n}} + \frac{1}{2}\cdot\frac{4t^2}{2n} + \frac{1}{3}\left(\frac{2t}{\sqrt{2n}}\right)^3 + \frac{1}{4}\left(\frac{2t}{\sqrt{2n}}\right)^4 + \cdots\right]$$

$$= \frac{-nt}{\sqrt{2n}} + \frac{nt}{\sqrt{2n}} + \frac{t^2}{2} + \frac{1}{3}\frac{n}{2}\left(\frac{2t}{\sqrt{2n}}\right)^3 + \frac{1}{4}\frac{n}{2}\left(\frac{2t}{\sqrt{2n}}\right)^4 + \cdots$$

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proceeding to the limit as $n \to \infty$ we have

$$\lim_{n \to \infty} \log M_Z(t) = t^2/2$$

$$\log \lim_{n \to \infty} M_Z(t) = t^2/2$$

$$M_Z(t) = e^{t^2/2}$$

which is m.g.f of a standard Normal Variate, showing that $Z = \frac{\chi^2 - n}{\sqrt{2n}}$ is a standard Normal Variate for $n \to \infty$.

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Problem:- 91, 95 If χ_1^2 & χ_2^2 are two independent χ^2-Variates with n_1, n_2 d.f respectively then

i) $\frac{\chi_1^2}{\chi_2^2}$ is $\beta_2\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$ Variate.

ii) $\frac{\chi_1^2}{\chi_1^2 + \chi_2^2}$ is $\beta_1\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$ Variate.

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Solution:- (i) Using Compound Probability Theorem, the joint probability differentials of χ_1^2 & χ_2^2 is given by,

$$dP(\chi_1^2, \chi_2^2) = dP_1(\chi_1^2)\, dP_2(\chi_2^2)$$

χ_1^2 & χ_2^2 are being independent.

$$= \frac{1}{2^{\frac{n_1}{2}} \Gamma n_1/2} e^{-\chi_1^2/2} (\chi_1^2)^{\frac{n_1}{2}-1} d\chi_1^2 \cdot \frac{1}{2^{n_2/2} \Gamma n_2/2} e^{-\chi_2^2/2} (\chi_2^2)^{\frac{n_2}{2}-1} d\chi_2^2$$

$$= \frac{1}{2^{\frac{n_1+n_2}{2}} \Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2}} e^{-\frac{1}{2}(\chi_1^2+\chi_2^2)} (\chi_2^2)^{\frac{n_2}{2}-1} (\chi_1^2)^{\frac{n_1}{2}-1} d\chi_1^2 d\chi_2^2$$

$$0 \le \chi_1^2, \chi_2^2 \le \infty$$

let $u = \dfrac{\chi_1^2}{\chi_2^2}$ & $v = \chi_2^2$

$$\chi_1^2 = \chi_2^2 u = uv \qquad \because \chi_2^2 = v$$

So that Jacobian of transformation

$$J = \frac{\partial(\chi_1^2, \chi_2^2)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial \chi_1^2}{\partial u} & \partial\chi_1^2/\partial v \\ \dfrac{\partial \chi_2^2}{\partial u} & \dfrac{\partial \chi_2^2}{\partial v} \end{vmatrix}$$

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$$= \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$$

So, that joint distribution of random variables U & V is given by,

$$dG(u,v) = \frac{1}{2^{\frac{n_1+n_2}{2}} \Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2}} e^{-\frac{v}{2}(u+1)} (uv)^{\frac{n_1}{2}-1} (v)^{\frac{n_2}{2}-1} \, v\,du\,dv$$

$$= \frac{1}{2^{\frac{n_1+n_2}{2}} \Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2}} e^{-\frac{v}{2}(u+1)} (u)^{\frac{n_1}{2}-1} (v)^{\frac{n_1+n_2}{2}-1} \, du\,dv$$

$$0 \leq u, v \leq \infty$$

Therefor the marginal distribution of u

$$\int_0^\infty dG(u,v) = \frac{(u)^{\frac{n_1}{2}-1} \, du}{2^{\frac{n_1+n}{2}} \Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2}} \int_0^\infty e^{-\frac{v}{2}(u+1)} v^{\frac{n_1+n_2}{2}-1} \, dv$$

As we know that

$$\int_0^\infty e^{-ay} y^{n-1} dy = \frac{\Gamma(n)}{a^n}$$

So, we have

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$$= \frac{(u)^{n_1/2-1}\,du}{2^{\frac{n_1+n_2}{2}}\,\Gamma n_1/2\,\Gamma n_2/2}\;\frac{\Gamma\frac{n_1+n_2}{2}}{\left(\frac{1+u}{2}\right)^{\frac{n_1+n_2}{2}}}$$

$$= \frac{2^{\frac{n_1+n_2}{2}}}{2^{\frac{n_1+n_2}{2}}\,\Gamma n_1/2\,\Gamma n_2/2}\;\Gamma\left(\frac{n_1+n_2}{2}\right)\frac{(u)^{n_1/2-1}}{(1+u)^{\frac{n_1+n_2}{2}}}\,du$$

$$= \frac{\Gamma\frac{n_1+n_2}{2}}{\Gamma n_1/2\;\Gamma n_2/2}\;\frac{(u)^{n_1/2-1}}{(1+u)^{\frac{n_1+n_2}{2}}}\,du$$

As we know,

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

So we have

$$= \frac{1}{\beta(n_1/2,\, n_2/2)}\;\frac{(u)^{n_1/2-1}}{(1+u)^{\frac{n_1+n_2}{2}}}\,du \qquad 0 \le u < \infty$$

which follows that $u = \chi_1^2/\chi_2^2$ is $\beta_2(n_1/2, n_1/2)$ variate.

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ii) The joint distribution of χ_1^2 & χ_2^2 is

$$dP(\chi_1^2, \chi_2^2) = \frac{1}{2^{\frac{n_1+n_2}{2}} \Gamma\frac{n_1}{2} \Gamma n_2/2} e^{-\frac{1}{2}(\chi_1^2+\chi_2^2)} (\chi_1^2)^{n_1/2-1} (\chi_2^2)^{n_2/2-1} d\chi_1^2 d\chi_2^2$$

$$0 \le \chi_1^2, \chi_2^2 \le \infty$$

let

$$U = \frac{\chi_1^2}{\chi_1^2+\chi_2^2}, \quad V = \chi_1^2 + \chi_2^2$$

$$\chi_1^2 = UV, \quad \chi_2^2 = V - \chi_1^2 = V - UV$$

$$\chi_1^2 = UV, \quad \chi_2^2 = V(1-U)$$

so, that the Jacobian of Transformation,

$$J = \frac{\partial(\chi_1^2, \chi_2^2)}{\partial(U,V)} = \begin{vmatrix} \frac{\partial\chi_1^2}{\partial U} & \frac{\partial\chi_1^2}{\partial V} \\ \frac{\partial\chi_2^2}{\partial U} & \frac{\partial\chi_2^2}{\partial V} \end{vmatrix}$$

$$= \begin{vmatrix} V & U \\ -V & 1-U \end{vmatrix}$$

$$= V - \cancel{UV} + \cancel{UV}$$

$$J = V$$

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$\therefore$

$$dG_1(U,V) = \frac{1}{2^{\frac{n_1+n_2}{2}}\Gamma\frac{n_1}{2}\Gamma\frac{n_2}{2}} e^{-V/2} (UV)^{n_1/2-1}[(1-U)V]^{n_2/2-1} V\,du\,dv$$

$$dG_1(U,V) = \frac{1}{2^{\frac{n_1+n_2}{2}}\Gamma\frac{n_1}{2}\Gamma\frac{n_2}{2}} e^{-V/2} (U)^{n_1/2-1}(1-U)^{n_2/2-1} V^{\left(\frac{n_1+n_2}{2}-1\right)} du\,dv$$

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$$\Rightarrow \quad dG_1(U) = \int_0^\infty dG_1(U,V) :$$

$$d(G_1(U)) = \frac{U^{n_1/2-1}(1-U)^{n_2/2-1}\,du}{2^{\frac{n_1+n_2}{2}}\Gamma\frac{n_1}{2}\Gamma\frac{n_2}{2}} \int_0^\infty e^{-V/2} V^{\frac{n_1+n_2}{2}-1} dV$$

$$d(G_1(U)) = \frac{U^{n_1/2-1}(1-U)^{n_2/2-1}\,du}{2^{\frac{n_1+n_2}{2}}\Gamma\frac{n_1}{2}\Gamma\frac{n_2}{2}} \frac{\Gamma\frac{(n_1+n_2)}{2}}{\left(\frac{1}{2}\right)^{\frac{n_1+n_2}{2}}}$$

$$d(G_1(U)) = \frac{2^{\frac{n_1+n_2}{2}}\Gamma\frac{n_1+n_2}{2}}{2^{\frac{n_1+n_2}{2}}\Gamma\frac{n_1}{2}\Gamma\frac{n_2}{2}} U^{(n_1/2-1)}(1-U)^{\frac{n_2}{2}-1} du$$

$$0 \le U \le \infty$$

$$dG_1(U) = \frac{1}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)} U^{\frac{n_1}{2}-1}(1-U)^{n_2/2-1} du$$

$$0 \le U \le \infty$$

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The Marginal distribution of v (Extra)

$$dG_2(v) = \int_0^\infty d\cdot G(u,v)$$

$$= \frac{1\cdot e^{-v/2}(v)^{\frac{n_1+n_2}{2}-1}}{2^{\frac{n_1+n_2}{2}}\Gamma n_1/2\,\Gamma n_2/2}\,dv\int_0^\infty u^{\frac{n_1}{2}-1}(1-u)^{\frac{n_2}{2}-1}\,du$$

$$dG_2(v) = \frac{e^{-v/2}(v)^{\frac{n_1+n_2}{2}-1}\,dv}{2^{\frac{n_1+n_2}{2}}\Gamma n_1/2\,\Gamma n_2/2}\,\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)$$

As we know,

$$\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right) = \frac{\Gamma(n_1/2)\,\Gamma(n_2/2)}{\Gamma\left(\frac{n_1+n_2}{2}\right)}$$

$$dG_2(v) = \frac{e^{-v/2}(v)^{\frac{n_1+n_2}{2}-1}\,dv}{2^{\frac{n_1+n_2}{2}}\Gamma(n_1/2)\,\Gamma(n_2/2)}\cdot\frac{\Gamma n_1/2\,\Gamma n_2/2}{\Gamma\left(\frac{n_1+n_2}{2}\right)}$$

$$dG_2(v) = \frac{1}{2^{\frac{n_1+n_2}{2}}\Gamma\left(\frac{n_1+n_2}{2}\right)}e^{-v/2}(v)^{\frac{n_1+n_2}{2}-1}\,dv \quad 0\le v<\infty$$

showing that u is $\beta_1\left(\frac{n_1}{2},\frac{n_2}{2}\right)$ variate & v is χ^2- variate with (n_1+n_2) d.f.

Student's t-distribution

Problem:- Derive the Student's t-distribution.

Solution:- Let W denote a r.v that N(0,1). Let V denote a r.v that is $\chi^2_{(n)}$ & let W & V be independent. Define a new random variable T where

$$T = \frac{W}{\sqrt{\frac{V}{n}}}$$

The joint probability density function of W and V is the product of the probability density functions of W & V.

$$f(w, v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}w^2} \frac{1}{2^{n/2}\Gamma n/2} v^{\frac{n}{2}-1} e^{-v/2} \qquad \begin{matrix} -\infty < w < \infty \\ 0 < v < \infty \end{matrix}$$

The change of variable technique will be used to find the probability density function of T

Let

$$t = \frac{w}{\sqrt{v/n}} \quad \& \quad v = u$$

$$\Rightarrow \quad w = t\sqrt{\frac{v}{n}} = t\sqrt{\frac{u}{n}}$$

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$$J = \begin{vmatrix} \frac{\partial v}{\partial u} & \frac{\partial w}{\partial u} \\ \frac{\partial v}{\partial t} & \frac{\partial w}{\partial t} \end{vmatrix} = \begin{vmatrix} 1 & \frac{1}{2}\frac{t}{\sqrt{u}}\, u^{1/2} \\ 0 & \sqrt{\frac{u}{n}} \end{vmatrix}$$

$$J = \sqrt{\frac{u}{n}}$$

The joint Probability density function of t & u is

$$g(t,u) = \frac{1}{\sqrt{2\pi}\; 2^{n/2}\,\Gamma n/2}\, u^{\frac{n}{2}-1} e^{-u/2}\, e^{-\frac{1}{2}t^2\frac{u}{n}} \cdot \sqrt{\frac{u}{n}}$$

$$0 \le u \le \infty$$
$$-\infty \le t \le \infty$$

$$g(t,u) = \frac{1}{\sqrt{2\pi}\; 2^{n/2}\,\Gamma n/2\,\sqrt{n}}\, u^{n/2-1/2}\, e^{-u/2\left(1+\frac{t^2}{n}\right)}$$

$$g_1(t) = \frac{1}{\sqrt{2\pi}\; 2^{n/2}\,\Gamma n/2\,\sqrt{n}} \int_0^\infty u^{\frac{n-1}{2}}\, e^{-u/2\left(1+\frac{t^2}{n}\right)}\, du$$

Put $\frac{u}{2}\left(1+\frac{t^2}{n}\right) = z$

$$u = \frac{2z}{1+t^2/n}$$

$$du = \frac{2}{1+t^2/n}\, dz$$

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$$g_1(t) = \frac{1}{\sqrt{2\pi}\, 2^{n/2} \Gamma\frac{n}{2} \sqrt{n}} \int_0^\infty \left(\frac{2z}{1+\frac{t^2}{n}}\right)^{\frac{n-1}{2}} e^{-z} \left(\frac{2}{1+t^2/n}\right) dz$$

$$g_1(t) = \frac{1}{\sqrt{2\pi}\, 2^{n/2} \Gamma\frac{n}{2} \sqrt{n}} \left(\frac{2}{1+\frac{t^2}{n}}\right)^{\frac{n-1}{2}} \left(\frac{2}{1+\frac{t^2}{n}}\right) \int_0^\infty z^{\frac{n-1}{2}} e^{-z} dz$$

$$g_1(t) = \frac{1}{\sqrt{2\pi}\, 2^{n/2} \Gamma\frac{n}{2} \sqrt{n}} \left(\frac{2}{1+\frac{t^2}{n}}\right)^{\frac{n+1}{2}} \int_0^\infty e^{-z} z^{\frac{n+1}{2}-1} dz$$

$$= \frac{1}{\sqrt{2\pi}\, 2^{n/2} \Gamma\frac{n}{2} \sqrt{n}} \frac{2^{\frac{n+1}{2}}}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} \Gamma\left(\frac{n+1}{2}\right)$$

$$= \frac{1}{2^{\frac{n+1}{2}} \sqrt{\pi} \sqrt{n}\, \Gamma\frac{n}{2}} \frac{2^{\frac{n+1}{2}}}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} \Gamma\left(\frac{n+1}{2}\right)$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\frac{1}{2}\, \Gamma\frac{n}{2} \sqrt{n}} \left(1+\frac{t^2}{n}\right)^{-\left(\frac{n+1}{2}\right)} \quad \because \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

$$= \frac{1}{\left(\frac{\Gamma\frac{1}{2}\, \Gamma\frac{n}{2}}{\Gamma\left(\frac{1}{2}+\frac{n}{2}\right)}\right)\sqrt{n}} \left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}}$$

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$$g_1(t) = \frac{1}{\beta(\frac{1}{2}, \frac{n}{2})\sqrt{n}} \left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}} \qquad -\infty \le t \le \infty$$

which is Probability density function of t- distribution.

Problem:- Show that total area under the curve of t-distribution is unity.

Solution:- As

$$g(t) = \frac{1}{\beta(\frac{1}{2}, \frac{n}{2})\sqrt{n}} \left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}} \qquad -\infty \le t \le \infty$$

$$\text{Area} = \int_{-\infty}^{\infty} g(t)\,dt = \int_{-\infty}^{\infty} \frac{1}{\beta(\frac{1}{2}, \frac{n}{2})\sqrt{n}} \left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}} dt$$

$$= \frac{1}{\beta(\frac{1}{2}, \frac{n}{2})\sqrt{n}} \int_{-\infty}^{\infty} \left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}} dt$$

As t- distribution is Symmetrical

So,

$$\text{Area} = \frac{2}{\beta(\frac{1}{2}, \frac{n}{2})\sqrt{n}} \int_{0}^{\infty} \left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}} dt$$

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Put $1 + \frac{t^2}{n} = \frac{1}{y} \Rightarrow t = \sqrt{n\left(\frac{1-y}{y}\right)}$

$$y = \left(1 + \frac{t^2}{n}\right)^{-1}$$

$$dy = -\left(1 + \frac{t^2}{n}\right)^{-2} \cdot \frac{2t}{n}\, dt$$

$$dy = -\left(\frac{1}{y}\right)^{-2} \frac{2}{n} \sqrt{n\left(\frac{1-y}{y}\right)}\, dt$$

$$\Rightarrow \quad dt = \frac{-n}{2\sqrt{n\left(\frac{1-y}{y}\right)}} \left(\frac{1}{y}\right)^2 dy$$

A.S $t = 0 \Rightarrow y = 1$

A.S $t = \infty \Rightarrow y = 0$

$$\Rightarrow \text{Area} = \frac{2}{\beta\left(\frac{1}{2}, \frac{n}{2}\right)\sqrt{n}} \int_1^0 y^{\frac{n+1}{2}} \frac{-n}{2\sqrt{n\left(\frac{1-y}{y}\right)}} \left(\frac{1}{y}\right)^2 dy$$

$$\text{Area} = \frac{2\, n}{\beta\left(\frac{1}{2}, \frac{n}{2}\right)\sqrt{n}\sqrt{n}\, 2} \int_0^1 y^{\frac{n+1}{2}} \frac{1}{\left(\frac{1-y}{y}\right)^{1/2}} \left(\frac{1}{y}\right)^2 dy$$

$$\text{Area} = \frac{1}{\beta\left(\frac{1}{2}, \frac{n}{2}\right)} \int_0^1 y^{\frac{n+1}{2} + \frac{1}{2} - 2} (1-y)^{-1/2}\, dy$$

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$$\text{Area} = \frac{1}{\beta(\frac{1}{2}, \frac{n}{2})} \int_0^1 y^{n/2 - 1} (1-y)^{1/2 - 1} \, dy$$

$$\text{Area} = \frac{1}{\beta(\frac{1}{2}, \frac{n}{2})} \cdot \beta(\frac{1}{2}, \frac{n}{2})$$

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$$\text{Area} = 1$$

$\Rightarrow$ Hence total Area under the t-curve is unity.

As Required.

Problem) Find the moments of t-distribution & calculate β & γ co-efficients.

Solution:- As

$$g(t) = \frac{1}{\beta(\frac{1}{2}, \frac{n}{2}) \sqrt{n}} \left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}} \quad -\infty \leq t \leq \infty$$

g(t) being symmetrical about t = 0, the odd order moments about origin vanishes

i.e,

$$\mu'_{2r+1} = 0 \quad ; r = 0, 1, 2, 3, \ldots$$

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The moments of even order about origin are

$$\mu'_{2r} = E(t^{2r}) = \int_{-\infty}^{\infty} t^{2r}\, g(t)\, dt$$

$$= 2\int_{0}^{\infty} t^{2r}\, g(t)\, dt$$

$$= 2\int_{0}^{\infty} t^{2r} \cdot \frac{1}{\beta(\frac{1}{2},\frac{n}{2})\sqrt{n}\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}\, dt$$

$$= \frac{2}{\beta(\frac{1}{2},\frac{n}{2})\sqrt{n}} \int_{0}^{\infty} \frac{t^{2r}}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}\, dt$$

put $1+\frac{t^2}{n} = \frac{1}{z} \Rightarrow t = \sqrt{\frac{n-nz}{z}}$

$$t^2 = \frac{n}{z} - n$$

$$2t\,dt = \frac{-n}{z^2}\, dz$$

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$$2\sqrt{\frac{n-nz}{z}}\,dt = \frac{-n}{z^2}\,dz$$

$$dt = \frac{-n\,dz}{2\sqrt{\frac{n-nz}{z}}\cdot z^2}$$

As $t=0 \Rightarrow z=1$

As $t=\infty \Rightarrow z=0$

so

$$\mu'_{2r} = \frac{\cancel{2}}{\beta(\frac{1}{2},\frac{n}{2})\sqrt{n}} \int_1^0 \frac{\left(\sqrt{\frac{n-nz}{z}}\right)^{2r}}{\left(\frac{1}{z}\right)^{\frac{n+1}{2}}} \frac{-n\,dz}{\cancel{2}\sqrt{\frac{n-nz}{z}}\cdot z^2}$$

$$= \frac{1}{\beta(\frac{1}{2},\frac{n}{2})\sqrt{n}} \int_0^1 \left(\sqrt{\frac{n-nz}{z}}\right)^{2r-1} \frac{z^{\frac{n+1}{2}}\,n\,dz}{z^2}$$

$$= \frac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})} \int_0^1 \frac{(n-nz)^{\frac{2r-1}{2}}}{(z)^{\frac{2r-1}{2}}}\, z^{\frac{n+1}{2}-2}\, n\,dz$$

$$= \frac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})} \int_0^1 (n-nz)^{\frac{2r-1}{2}}\, z^{\frac{n+1}{2}-2-\frac{2r-1}{2}}\, n\,dz$$

$$= \frac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})} \int_0^1 n^{\frac{2r-1}{2}} (1-z)^{\frac{2r-1}{2}}\, z^{\frac{n+1-4-2r+1}{2}}\, n\,dz$$

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$$= \frac{1}{\sqrt{n}\,\beta\left(\frac{1}{2},\frac{n}{2}\right)} \int_0^1 n^{\frac{2r}{2}-\frac{1}{2}+1} (1-z)^{r-\frac{1}{2}} (z)^{\frac{n-2r-2}{2}} dz$$

$$= \frac{1}{\sqrt{n}\,\beta\left(\frac{1}{2},\frac{n}{2}\right)} \int_0^1 n^{r-\frac{1}{2}+1} z^{\frac{n}{2}-r-1} (1-z)^{r-\frac{1}{2}} dz$$

$$= \frac{n^r \sqrt{n}}{\sqrt{n}\,\beta\left(\frac{1}{2},\frac{n}{2}\right)} \int_0^1 z^{\left(\frac{n}{2}-r\right)-1} (1-z)^{\left(r+\frac{1}{2}\right)-1} dz$$

$$= \frac{n^r}{\beta\left(\frac{1}{2},\frac{n}{2}\right)} \beta\left(\frac{n}{2}-r,\ r+\frac{1}{2}\right) \longrightarrow (A)$$

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Since $\mu'_{2r+1} = 0$

$$\Rightarrow \quad \mu'_{2r} = \mu_{2r}$$

$$\mu_{2r} = \frac{n^r}{\beta\left(\frac{1}{2},\frac{n}{2}\right)} \beta\left(\frac{n}{2}-r,\ r+\frac{1}{2}\right) \longrightarrow (B)$$

Put $r = 1$

$$\mu_2 = \frac{n}{\beta\left(\frac{1}{2}, \frac{n}{2}\right)} \beta\left(\frac{n}{2}-1, \frac{3}{2}\right)$$

$$\mu_2 = \frac{n}{\dfrac{\Gamma\frac{1}{2}\,\Gamma\frac{n}{2}}{\Gamma\frac{n+1}{2}}} \cdot \frac{\Gamma\left(\frac{n}{2}-1\right)\Gamma\frac{3}{2}}{\Gamma\frac{n+1}{2}}$$

$$\mu_2 = \frac{n}{\Gamma\frac{1}{2}\,\Gamma\frac{n}{2}} \cdot \Gamma\left(\frac{n}{2}-1\right)\Gamma\frac{3}{2}$$

$$\mu_2 = \frac{n\,\Gamma\left(\frac{n}{2}-1\right)\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma\frac{1}{2}\left(\frac{n}{2}-1\right)\Gamma\left(\frac{n}{2}-1\right)}$$

$$\mu_2 = \frac{n}{2\left(\frac{n-2}{2}\right)}$$

$$\mu_2 = \frac{n}{n-2}$$

put $r = 2$

we get,

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$$\mu_4 = \frac{n^2}{\beta\left(\frac{1}{2}, \frac{n}{2}\right)} \beta\left(\frac{n}{2}-2, 5/2\right)$$

$$\mu_4 = \frac{n^2}{\frac{\Gamma 1/2 \cdot \Gamma n/2}{\Gamma \frac{n+1}{2}}} \cdot \frac{\Gamma n/2 - 2 \; \Gamma 5/2}{\Gamma \frac{n+1}{2}}$$

$$\Gamma\left(\frac{n}{2}-2\right) \; 3/2 \cdot 1/2 \; \Gamma 1/2$$

$$\mu_4 = \frac{n^2}{\Gamma 1/2 \, (n/2 - 1)(n/2 - 2)\, \Gamma(n/2 - 2)}$$

$$\mu_4 = \frac{3 \cdot n^2}{4\left(\frac{n-2}{2}\right)\left(\frac{n-4}{2}\right)}$$

$$\mu_4 = \frac{3n^2}{(n-2)(n-4)}$$

Now, β- Co-efficents are,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0 \qquad \because \mu_3 = 0$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

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$$\beta_2 = \frac{3n^2}{(n-2)(n-4)} \times \left(\frac{n-2}{n}\right)^2$$

$$\beta_2 = 3\left(\frac{n-2}{n-4}\right)$$

Now, Gamma (γ) Co-efficients are

$$\gamma_1 = \sqrt{\beta_1} = 0$$

$$\gamma_2 = \beta_2 - 3$$

$$\gamma_2 = 3\left(\frac{n-2}{n-4}\right) - 3$$

$$\gamma_2 = \frac{3(n-2) - 3(n-4)}{n-4}$$

$$\gamma_2 = \frac{3n - 6 - 3n + 12}{n-4}$$

$$\gamma_2 = \frac{6}{n-4}$$

As Required.

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Problem:- Limiting Form of t-distribution
OR
Show that when $n \to \infty$ the p.d.f of t-distribution with n degree of freedom
i.e,

$$f(t) = \frac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} \longrightarrow \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \; ; -\infty < t < \infty$$

Solution:-

$$f(t) = \frac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})} \cdot \frac{1}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}$$

$$\lim_{n\to\infty} f(t) = \lim_{n\to\infty} \frac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})} \cdot \lim_{n\to\infty} \frac{1}{\left(1+t^2/n\right)^{\frac{n+1}{2}}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}\,\frac{\Gamma\frac{1}{2}\,\Gamma\frac{n}{2}}{\Gamma\frac{n+1}{2}}} \cdot \lim_{n\to\infty} \frac{1}{\left[\left(1+t^2/n\right)^{n}\right]^{1/2} \left(1+\frac{t^2}{n}\right)^{1/2}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}\,.\sqrt{\pi}} \; \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\frac{n}{2}} \cdot \lim_{n\to\infty} \frac{1}{\left[\left(1+\frac{t^2}{n}\right)^{n}\right]^{1/2} \left(1+\frac{t^2}{n}\right)^{1/2}} \longrightarrow (A)$$

As we know that

$$\lim_{n\to\infty} \frac{\lceil n+k}{\lceil n} = n^k$$

$$\lim_{n\to\infty} \left(1+\frac{t^2}{n}\right)^n = e^{t^2}$$

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So A $\Rightarrow$

$$\lim_{n\to\infty} f(t) = \lim_{n\to\infty} \left(\frac{1}{\sqrt{n\pi}}\right)\left(\frac{n}{2}\right)^{1/2} \cdot \frac{1}{(e^{t^2})^{1/2} \cdot 1}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{2}\cdot\sqrt{\pi}\cdot\sqrt{n}} \cdot \frac{1}{e^{+t^2/2}}$$

$$= \frac{e^{-t^2/2}}{\sqrt{2\pi}}$$

$\Rightarrow$

$$f(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \quad \text{as } n \to \infty$$

As Required.

Problem:- Show that for t-distribution with n d.f mean deviation about the mean is given by

$$\frac{\sqrt{n}\,\Gamma\frac{n-1}{2}}{\sqrt{\pi}\,\Gamma n/2}$$

Solution:- $f(t) = \frac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})} \cdot \frac{1}{(1+\frac{t^2}{n})^{\frac{n+1}{2}}}$; $-\infty \le t \le \infty$

$M.D = E|t-\bar{t}|$

Here $\bar{t} = \text{Mean} = \mu_1' = 0$

$\Rightarrow$

$$M.D = E|t| = \int_{-\infty}^{\infty} |t|\, f(t)\, dt$$

$$= \int_{-\infty}^{\infty} |t| \cdot \frac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})} \cdot \frac{1}{(1+\frac{t^2}{n})^{\frac{n+1}{2}}}\, dt$$

$$= \frac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})} \int_{-\infty}^{\infty} \frac{|t|}{(1+\frac{t^2}{n})^{\frac{n+1}{2}}}\, dt$$

$$\Rightarrow M.D = \frac{1}{\sqrt{n}\,\beta\left(\frac{1}{2},\frac{n}{2}\right)} 2\int_0^\infty \frac{t}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} dt \rightarrow (1)$$

Put $1+\frac{t^2}{n} = \frac{1}{y}$

$$\frac{n+t^2}{n} = \frac{1}{y}$$

$$n+t^2 = \frac{n}{y}$$

$$t^2 = \frac{n}{y} - n$$

$$2t\,dt = \frac{-n}{y^2}\,dy$$

As $t=0 \Rightarrow y=1$

$t=\infty \Rightarrow y=0$

So Eq (1) $\Rightarrow$

$$M.D = \frac{2}{\sqrt{n}\,\beta\left(\frac{1}{2},\frac{n}{2}\right)} \int_1^0 \frac{-n/y^2\,dy}{2\left(\frac{1}{y}\right)^{\frac{n+1}{2}}}$$

$$M.D = \frac{n}{\sqrt{n}\,\beta\left(\frac{1}{2},\frac{n}{2}\right)} \int_0^1 \frac{1}{y^2}\, y^{\frac{n+1}{2}}\,dy$$

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$$= \frac{\sqrt{n}}{\beta\left(\frac{1}{2},\frac{n}{2}\right)} \int_0^1 y^{\frac{n+1}{2}-2}\,dy$$

$$= \frac{\sqrt{n}}{\beta\left(\frac{1}{2},\frac{n}{2}\right)} \int_0^1 y^{\frac{n-3}{2}}\,dy$$

$$= \frac{\sqrt{n}}{\beta\left(\frac{1}{2},\frac{n}{2}\right)} \int_0^1 (y)^{\left(\frac{n-1}{2}\right)-1} (1-y)^{1-1}\,dy$$

$$= \frac{\sqrt{n}}{\beta\left(\frac{1}{2},\frac{n}{2}\right)} \beta\left(\frac{n-1}{2}, 1\right)$$

$$= \frac{\sqrt{n}}{\beta\left(\frac{1}{2},\frac{n}{2}\right)} \beta\left(\frac{n-1}{2}, 1\right)$$

$$= \frac{\sqrt{n}}{\dfrac{\Gamma\frac{1}{2}\,\Gamma\frac{n}{2}}{\Gamma\frac{n+1}{2}}} \cdot \frac{\Gamma\frac{n-1}{2}\cdot\Gamma 1}{\Gamma\frac{n+1}{2}}$$

$$= \frac{\sqrt{n}}{\sqrt{\pi}} \frac{\Gamma\frac{n-1}{2}}{\Gamma\frac{n}{2}} \qquad \because \Gamma\tfrac{1}{2} = \sqrt{\pi}, \quad \Gamma 1 = 1$$

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$\Rightarrow M.D = \dfrac{\sqrt{n}\,\Gamma\frac{n-1}{2}}{\sqrt{\pi}\,\Gamma\frac{n}{2}}$

As Required.

91, 92.

Problem:- If random variates X_1 & X_2 are independent & follows χ^2 distribution with n d.f. Show that $\dfrac{\sqrt{n}\,(X_1 - X_2)}{2\sqrt{X_1 X_2}}$

is distributed as Student's t distribution with n d.f independently of $X_1 + X_2$.

Solution:- we have

$$P(X_1, X_2) = \frac{1}{2^{n/2}\,\Gamma\frac{n}{2}}\, e^{-\frac{x_1}{2}} (x_1)^{n/2 - 1} \cdot \frac{1}{2^{n/2}\,\Gamma\frac{n}{2}}\, e^{-\frac{x_2}{2}} (x_2)^{n/2 - 1}$$

$$0 \le x_1 \le \infty, \quad 0 \le x_2 \le \infty$$

$$P(X_1, X_2) = \frac{1}{\left[2^{n/2}\right]^2 \left[\Gamma\frac{n}{2}\right]^2}\, e^{-\frac{1}{2}(x_1 + x_2)} (x_1)^{n/2 - 1} (x_2)^{n/2 - 1}$$

putting $u = \dfrac{\sqrt{n}\,(x_1 - x_2)}{2\sqrt{x_1 x_2}}$ & $v = x_1 + x_2$

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i.e, $\frac{u}{\sqrt{n}} = \frac{x_1 - x_2}{2\sqrt{x_1 x_2}}$

$$\frac{2\sqrt{x_1 x_2}}{x_1 - x_2} = \frac{\sqrt{n}}{u}$$

$$\frac{4(x_1 x_2)}{(x_1 - x_2)^2} = \frac{n}{u^2}$$

$$1 + \frac{4 x_1 x_2}{(x_1 - x_2)^2} = 1 + \frac{n}{u^2}$$

$$\frac{x_1^2 + x_2^2 - 2x_1 x_2 + 4x_1 x_2}{(x_1 - x_2)^2} = 1 + \frac{n}{u^2}$$

$$\frac{(x_1 + x_2)^2}{(x_1 - x_2)^2} = 1 + \frac{n}{u^2}$$

$$\frac{v^2}{(x_1 - x_2)^2} = 1 + \frac{n}{u^2}$$

$$\frac{v}{x_1 - x_2} = \sqrt{1 + \frac{n}{u^2}}$$

$$x_1 - x_2 = v \cdot \frac{1}{\sqrt{1 + \frac{n}{u^2}}}$$

Also, $x_1 + x_2 = v$

Adding above Eq we get,

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$$2x_1 = v\left[1 + \frac{1}{\sqrt{1+\frac{n}{u^2}}}\right]$$

$$x_1 = \frac{v}{2}\left[1 + \frac{1}{\sqrt{1+\frac{n}{u^2}}}\right]$$

Also,

$$x_2 = v - x_1$$

$$x_2 = v - \frac{v}{2}\left[1 + \frac{1}{\sqrt{1+\frac{n}{u^2}}}\right]$$

$$x_2 = \frac{v}{2} - \frac{v}{2} \cdot \frac{1}{\sqrt{1+\frac{n}{u^2}}}$$

$$x_2 = \frac{v}{2}\left[1 - \frac{1}{\sqrt{1+\frac{n}{u^2}}}\right]$$

Now,

$$x_1 = \frac{v}{2}\left[1 + \frac{u}{\sqrt{u^2+n}}\right] = \frac{v}{2} + \frac{uv}{2\sqrt{u^2+n}}$$

$$x_1 = \frac{v}{2} + \frac{1}{2} uv (u^2+n)^{-1/2}$$

$$x_2 = \frac{v}{2}\left[1 - \frac{u}{\sqrt{u^2+n}}\right] = \frac{v}{2} - \frac{uv}{2\sqrt{u^2+n}}$$

$$x_2 = \frac{v}{2} - \frac{1}{2} uv (u^2+n)^{-1/2}$$

Now,

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$$\frac{\partial x_1}{\partial u} = 0 + \frac{1}{2}\left[v(u^2+n)^{-1/2} + uv\left(\frac{-1}{2}\right)(u^2+n)^{-3/2}(2u)\right]$$

$$\frac{\partial x_1}{\partial u} = \frac{1}{2}\left[\frac{v}{\sqrt{u^2+n}} - \frac{u^2 v}{(u^2+n)^{3/2}}\right]$$

$$\frac{\partial x_1}{\partial u} = \frac{1}{2}\left[\frac{v(u^2+n) - u^2 v}{(u^2+n)^{3/2}}\right]$$

$$= \frac{1}{2}\left[\frac{u^2 v + vn - u^2 v}{(u^2+n)^{3/2}}\right]$$

$$\frac{\partial x_1}{\partial u} = \frac{1}{2}\left[\frac{nv}{(u^2+n)^{3/2}}\right]$$

$$\frac{\partial x_1}{\partial v} = \frac{1}{2} + \frac{1}{2}u(u^2+n)^{-1/2} = \frac{1}{2} + \frac{u}{2\sqrt{u^2+n}}$$

Similarly,

$$\frac{\partial x_2}{\partial u} = \frac{-1}{2}\left[\frac{nv}{(u^2+n)^{3/2}}\right]$$

$$\frac{\partial x_2}{\partial v} = \frac{1}{2} - \frac{u}{2\sqrt{u^2+n}}$$

we have,

$$J = \frac{\partial(x_1, x_2)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{nv}{2(u^2+n)^{3/2}} & \frac{1}{2} + \frac{u}{2\sqrt{u^2+n}} \\ \frac{-nv}{2(u^2+n)^{3/2}} & \frac{1}{2} - \frac{u}{2\sqrt{u^2+n}} \end{vmatrix}$$

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$$J = \left(\frac{nv}{2(u^2+n)^{3/2}}\right)\left(\frac{1}{2} - \frac{u}{2\sqrt{u^2+n}}\right) + \left(\frac{1}{2} + \frac{u}{2\sqrt{u^2+n}}\right)\left(\frac{nv}{2(u^2+n)^{3/2}}\right)$$

$$J = \frac{nv}{2(u^2+n)^{3/2}}\left[\frac{1}{2} - \frac{u}{2\sqrt{u^2+n}} + \frac{1}{2} + \frac{u}{2\sqrt{u^2+n}}\right]$$

$$J = \frac{nv}{2(u^2+n)^{3/2}} = \frac{1}{2} \cdot \frac{nv}{(n)^{3/2}\left[1 + \frac{u^2}{n}\right]^{3/2}}$$

$$J = \frac{v}{2\sqrt{n}\left[1 + \frac{u^2}{n}\right]^{3/2}}$$

$\therefore$ the joint probability density function of u, v is

$$g(u,v) = P(x_1, x_2)\,|J|$$

$$P(u,v) = \frac{e^{-\frac{1}{2}v}}{[2^{n/2}]^2[\Gamma n/2]^2}\left[\frac{v}{2}\left(1 + \frac{1}{\sqrt{1+\frac{n}{u^2}}}\right)\right]^{n/2-1} \times$$

$$\left[\frac{v}{2}\left(1 + \frac{1}{\sqrt{1+\frac{n}{u^2}}}\right)\right]^{n/2-1} \cdot \frac{v}{2\sqrt{n}\left(1 + \frac{u^2}{n}\right)^{3/2}}$$

$$-\infty < u < \infty, \quad 0 \leq v \leq \infty$$

$$P(u,v) = \frac{e^{-1/2\,v}}{2^n[\Gamma n/2]^2}\left(\frac{v}{2}\right)^{\frac{n}{2}-1+\frac{n}{2}-1+1}\left(1 + \frac{1}{\sqrt{1+\frac{n}{u^2}}}\right)^{\frac{n}{2}-1} \times$$

$$\left(1 - \frac{1}{\sqrt{1+\frac{n}{u^2}}}\right)^{n/2-1} \frac{1}{\sqrt{n}\left(1 + \frac{u^2}{n}\right)^{3/2}}$$

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$$P(u,v) = \frac{e^{-\frac{1}{2}v}}{2^{n}\sqrt{n}\left[\Gamma\frac{n}{2}\right]^{2}}\,\frac{(v)^{n-1}}{2^{n-1}}\left[1-\frac{1}{\left(1+\frac{n}{u^{2}}\right)}\right]^{\frac{n}{2}-1}\frac{1}{\left(1+\frac{u^{2}}{n}\right)^{3/2}}$$

$$P(u,v) = \frac{e^{-\frac{1}{2}v}}{2^{2n-1}\cdot\sqrt{n}\left[\Gamma\frac{n}{2}\right]^{2}}\,v^{n-1}\left[1-\frac{u^{2}}{u^{2}+n}\right]^{\frac{n}{2}-1}\left[\frac{1}{1+\frac{u^{2}}{n}}\right]^{3/2}$$

$$P(u,v) = \frac{e^{-\frac{1}{2}v}\,v^{n-1}}{2^{2n-1}\sqrt{n}\left[\Gamma\frac{n}{2}\right]^{2}}\left[\frac{n}{u^{2}+n}\right]^{\frac{n}{2}-1}\left[\frac{n}{u^{2}+n}\right]^{3/2}$$

$$P(u,v) = \frac{e^{-\frac{1}{2}v}\,v^{n-1}}{2^{2n-1}\sqrt{n}\left[\Gamma\frac{n}{2}\right]^{2}}\left[\frac{\cancel{n}}{\cancel{n}\left(1+\frac{u^{2}}{n}\right)}\right]^{\frac{n}{2}-1+\frac{3}{2}}$$

$$P(u,v) = \frac{e^{-\frac{1}{2}v}\,v^{n-1}}{2^{2n-1}\sqrt{n}\left[\Gamma\frac{n}{2}\right]^{2}}\cdot\frac{1}{\left(1+\frac{u^{2}}{n}\right)^{\frac{n}{2}+\frac{1}{2}}}$$

$$-\infty \le u \le \infty, \quad 0 \le v \le \infty$$

Applying legender duplication formula, i.e,

$$\Gamma(n) = \frac{2^{n-1}\,\Gamma\frac{n}{2}\,\Gamma\frac{n+1}{2}}{\sqrt{\pi}}$$

we have,

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$$\Gamma\frac{n}{2} = \frac{\sqrt{\pi}\,\Gamma(n)}{2^{n-1}\,\Gamma\frac{n+1}{2}}$$

$$2^{2n-1}\,\Gamma\tfrac{n}{2}\,\Gamma\tfrac{n}{2} = 2^{2n-1}\cdot\frac{\sqrt{\pi}\,\Gamma(n)}{2^{n-1}\,\Gamma\frac{n+1}{2}}\,\Gamma\tfrac{n}{2}$$

$$= 2^{n}\,\Gamma n\,\frac{\sqrt{\pi}\cdot\Gamma\frac{n}{2}}{\Gamma\frac{n+1}{2}}$$

$$= 2^{n}\,\Gamma n\,\frac{\Gamma\frac{1}{2}\,\Gamma\frac{n}{2}}{\Gamma\frac{n+1}{2}}$$

$$= 2^{n}\,\Gamma n\;\beta\left(\frac{1}{2},\frac{n}{2}\right)$$

$\therefore$

$$g(u,v) = \frac{1}{2^{n}\,\Gamma(n)\,\beta\left(\frac{1}{2},\frac{n}{2}\right)}\cdot\frac{e^{-\frac{1}{2}v}\,v^{n-1}}{\sqrt{n}}\;\frac{1}{\left(1+\frac{u^2}{n}\right)^{\frac{n+1}{2}}}$$

So we have,

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$$g(u,v) = \frac{1}{2^n \Gamma(n)} e^{-\frac{1}{2}v} v^{n-1} \frac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})} \frac{1}{\left(1+\frac{u^2}{n}\right)^{\frac{n+1}{2}}}$$

$-\infty \leq u \leq \infty$,

$0 \leq v \leq \infty$

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SNEDECOR'S F-Distribution

Problem:- Derive F- Distribution.

Proof:- Consider two independent chi-square random Variables U & V having n_1 & n_2 d.f respectively, the joint probability density function of U & V is,

$$h(U,V) = \frac{1}{2^{\frac{n_1}{2}}\Gamma\frac{n_1}{2}} U^{\frac{n_1}{2}-1} e^{-U/2} \cdot \frac{1}{2^{\frac{n_2}{2}}\Gamma\frac{n_2}{2}} V^{\frac{n_2}{2}-1} e^{-V/2}$$

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$0 \leq U \leq \infty,$
$0 \leq V < \infty$

$$h(U,V) = \frac{1}{2^{\frac{n_1+n_2}{2}}\Gamma\frac{n_1}{2}\Gamma\frac{n_2}{2}} e^{-\frac{U+V}{2}} U^{\frac{n_1}{2}-1} V^{\frac{n_2}{2}-1}$$

We define a new random Variable

$$F = \frac{U/n_1}{V/n_2}$$

We wish to find the Probability density Function of F.

let

$$F = \frac{U/n_1}{V/n_2} \qquad Z = V$$

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$$F = \frac{U}{n_1} \cdot \frac{n_2}{V} \Rightarrow U = \frac{F V n_1}{n_2}$$

$$U = \frac{F Z n_1}{n_2} \quad \because V = Z$$

The Jacobean of the transformation is,

$$J = \begin{vmatrix} \frac{\partial U}{\partial F} & \frac{\partial U}{\partial Z} \\ \frac{\partial V}{\partial F} & \frac{\partial V}{\partial Z} \end{vmatrix} = \begin{vmatrix} \frac{n_1}{n_2} Z & \frac{n_1}{n_2} F \\ 0 & 1 \end{vmatrix}$$

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$$J = \frac{n_1}{n_2} Z$$

The joint Probability density function of F & Z is,

$$g(F,Z) = \frac{1}{2^{\frac{n_1+n_2}{2}} \Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2}} e^{-\frac{1}{2}\left[\frac{n_1 F Z}{n_2} + Z\right]} \left(\frac{n_1 F Z}{n_2}\right)^{\frac{n_1}{2}-1} \times Z^{\frac{n_2}{2}-1} \frac{n_1 Z}{n_2}$$

$$g(F,Z) = \frac{1}{2^{\frac{n_1+n_2}{2}} \Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2}} e^{-Z/2\left[1+\frac{n_1}{n_2}F\right]} \left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} F^{\frac{n_1}{2}-1} Z^{\frac{n_1+n_2}{2}-1}$$

$$0 \leq Z \leq \infty$$

$$0 \leq F \leq \infty$$

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The marginal density function of F is

$$g_1(F) = \frac{1}{2^{\frac{n_1+n_2}{2}}\sqrt{\frac{n_1}{2}}\sqrt{n_2/2}}\left(\frac{n_1}{n_2}\right)^{n_1/2} F^{\frac{n_1}{2}-1}\int_0^\infty Z^{\frac{n_1+n_2}{2}-1} e^{-Z/2\left(1+\frac{n_1}{n_2}F\right)}\,dZ$$

Put $\frac{Z}{2}\left(1+\frac{n_1}{n_2}F\right) = y$

$$Z = \frac{2y}{1+\frac{n_1}{n_2}F} \Rightarrow dZ = \frac{2}{1+\frac{n_1}{n_2}F}\,dy$$

If $Z=0$ then $y=0$

If $Z=\infty$ then $y=\infty$

$$g_1(F) = \frac{1}{2^{\frac{n_1+n_2}{2}}\sqrt{\frac{n_1}{2}}\sqrt{n_2/2}}\left(\frac{n_1}{n_2}\right)^{n_1/2} F^{\frac{n_1}{2}-1}\int_0^\infty \left[\frac{2y}{\left(1+\frac{n_1}{n_2}F\right)}\right]^{\frac{n_1+n_2}{2}-1} e^{-y}\frac{2}{1+\frac{n_1}{n_2}F}\,dy$$

$$= \frac{1}{2^{\frac{n_1+n_2}{2}}\sqrt{\frac{n_1}{2}}\sqrt{\frac{n_2}{2}}}\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} F^{\frac{n_1}{2}-1}\frac{2^{\frac{n_1+n_2}{2}}}{\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}\int_0^\infty y^{\frac{n_1+n_2}{2}-1}e^{-y}\,dy$$

$$= \frac{1}{\sqrt{\frac{n_1}{2}}\sqrt{\frac{n_2}{2}}}\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} F^{\frac{n_1}{2}-1}\frac{1}{\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}\sqrt{\frac{n_1+n_2}{2}}$$

$$= \frac{\sqrt{\frac{n_1}{2}+\frac{n_2}{2}}}{\sqrt{\frac{n_1}{2}}\sqrt{n_2/2}}\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} F^{\frac{n_1}{2}-1}\left(\frac{n_2+n_1F}{n_2}\right)^{-\frac{n_1+n_2}{2}}$$

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$$= \frac{1}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} n_1^{n_1/2} n_2^{-n_1/2} F^{\frac{n_1}{2}-1} n_2^{\frac{n_1+n_2}{2}} (n_2+n_1F)^{-\frac{n_1+n_2}{2}}$$

$$= \frac{1}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} n_1^{n_1/2} n_2^{n_2/2} F^{\frac{n_1}{2}-1} (n_2+n_1F)^{-\frac{n_1+n_2}{2}}$$

$\Rightarrow$

$$g_1(F) = \frac{(n_1)^{\frac{n_1}{2}} (n_2)^{n_2/2}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \frac{F^{\frac{n_1}{2}-1}}{(n_2+n_1F)^{\frac{n_1+n_2}{2}}} \quad 0<F<\infty$$

which is density function with n_1 & n_2 degree of freedom, of F-Distribution.

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Problem) Find the Mode of F-Distribution.

Solution: As we know that

$$g(F) = \frac{(n_1)^{n_1/2} (n_2)^{n_2/2}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} F^{\frac{n_1}{2}-1} (n_2+n_1F)^{-\frac{n_1+n_2}{2}}$$

$$0<F<\infty$$

$$g'(F) = \frac{(n_1)^{n_1/2}(n_2)^{n_2/2}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \left[\frac{\left(\frac{n_1}{2} - F\right) F^{\frac{n_1}{2}-2}}{(n_2 + n_1 F)^{\frac{n_1+n_2}{2}}} - \left(\frac{n_1+n_2}{2}\right)(n_2 + n_1 F)^{-\frac{n_1+n_2}{2}-1} n_1 F^{\frac{n_1}{2}-1} \right]$$

put $g'(F) = 0$

$$\Rightarrow \frac{(n_1)^{n_1/2}(n_2)^{n_2/2}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \frac{F^{\frac{n_1}{2}-2}}{(n_2+n_1F)^{\frac{n_1+n_2}{2}+1}} \left[\frac{\frac{n_1}{2} - 1}{(n_2+n_1F)^{-1}} - n_1 \left(\frac{n_1+n_2}{2}\right) F \right] = 0$$

$$\Rightarrow \frac{\frac{n_1}{2} - 1}{(n_2+n_1F)^{-1}} - n_1\left(\frac{n_1+n_2}{2}\right)F = 0$$

$$\left(\frac{n_1}{2} - 1\right)(n_2 + n_1 F) = n_1\left(\frac{n_1+n_2}{2}\right)F$$

$$\left(\frac{n_1 - 2}{2}\right)(n_2 + n_1 F) = n_1\left(\frac{n_1+n_2}{2}\right)F$$

$$(n_1 - 2)(n_2 + n_1 F) = n_1(n_1+n_2)F$$

$$n_2(n_1 - 2) + (n_1 - 2)n_1 F = n_1(n_1 + n_2)F$$

$$n_2(n_1 - 2) = n_1(n_1+n_2)F - (n_1 - 2)n_1 F$$

$$n_2(n_1 - 2) = n_1 F(n_1 + n_2 - n_1 + 2)$$

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$$\Rightarrow \quad F = \frac{n_2(n_1-2)}{n_1(n_2+2)}$$

It may be varified $g''(F) < 0$ at

$$F = \frac{n_2(n_1-2)}{n_1(n_2+2)} \quad \text{provided } n_1 > 2$$

Hence

$$\text{Mode} = \frac{n_2(n_1-2)}{n_1(n_2+2)}$$

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Problem:- Find the value of μ_r' for the F-distribution and find Mean & Variance of the distribution.

Solution:- As for F-distribution, p.d.f is given by,

$$g(F) = \frac{n_1^{\frac{n_1}{2}} \, n_2^{n_2/2}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} F^{\frac{n_1}{2}-1} \left(n_2 + n_1 F\right)^{-\frac{n_1+n_2}{2}}$$

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$$\mu_r' = E(F^r) = \int_0^\infty F^r g(F) dF$$

$$\mu_r' = \int_0^\infty F^r \cdot \frac{(n_1)^{n_1/2} (n_2)^{n_2/2}}{\beta(\frac{n_1}{2}, \frac{n_2}{2})} F^{\frac{n_1}{2}-1} (n_2 + n_1 F)^{-\frac{n_1+n_2}{2}} dF$$

$$= \frac{(n_1)^{n_1/2} (n_2)^{n_2/2}}{\beta(\frac{n_1}{2}, \frac{n_2}{2})} \int_0^\infty F^{r+\frac{n_1}{2}-1} n_2^{-\frac{n_1+n_2}{2}} \left(1+\frac{n_1}{n_2}F\right)^{-\frac{n_1+n_2}{2}} dF$$

$$= \frac{(n_1)^{n_1/2} (n_2)^{n_2/2}}{\beta(\frac{n_1}{2}, \frac{n_2}{2})} n_2^{-\frac{n_1+n_2}{2}} \int_0^\infty F^{r+\frac{n_1}{2}-1} \left(1+\frac{n_1}{n_2}F\right)^{-\frac{n_1+n_2}{2}} dF$$

$$\mu_r' = \frac{(n_1)^{n_1/2} (n_2)^{-n_1/2}}{\beta(\frac{n_1}{2}, \frac{n_2}{2})} \int_0^\infty F^{r+\frac{n_1}{2}-1} \left(1+\frac{n_1}{2}F\right)^{-\frac{n_1+n_2}{2}} dF$$

put $1 + \frac{n_1}{n_2} F = \frac{1}{p}$

$$\frac{n_1 F}{n_2} = \frac{1}{p} - 1$$

$$F = \frac{n_2}{n_1}\left(\frac{1}{p} - 1\right)$$

$$dF = -\frac{n_2}{n_1} \frac{1}{p^2} dp$$

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If $F = 0$ then $P = 1$

If $F = \infty$ then $p = 0$

$$\mu_r' = \frac{n_1^{\frac{n_1}{2}}\, n_2^{-n_1/2}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \int_1^0 \left[\frac{n_2}{n_1}\left(\frac{1}{p}-1\right)\right]^{r+\frac{n_1}{2}-1} \left(\frac{1}{p}\right)^{-\frac{n_1+n_2}{2}} \left(-\frac{n_2}{n_1}\frac{1}{p^2}dp\right)$$

$$\mu_r' = (n_1)^{\frac{n_1}{2}} (n_2)^{-n_1/2} \left(\frac{n_2}{n_1}\right)^{r+\frac{n_1}{2}} \int_0^1 \left(\frac{1-p}{p}\right)^{r+\frac{n_1}{2}-1} p^{\frac{n_1+n_2}{2}-2}\, dp$$

$$\mu_r' = \frac{\left(\frac{n_2}{n_1}\right)^r}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \int_0^1 (1-p)^{r+\frac{n_1}{2}-1} p^{\frac{n_1+n_2}{2}-2-r-\frac{n_1}{2}+1}\, dp$$

$$\mu_r' = \frac{\left(\frac{n_2}{n_1}\right)^r}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \int_0^1 (1-p)^{r+\frac{n_1}{2}-1} p^{\frac{n_2-2r}{2}-1}\, dp$$

$$\mu_r' = \left(\frac{n_2}{n_1}\right)^r \frac{\beta\left(\frac{n_2}{2}-r, \frac{n_1}{2}+r\right)}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)}$$

$$\mu_r' = \left(\frac{n_2}{n_1}\right)^r \frac{\beta\left(\frac{n_1}{2}+r, \frac{n_2}{2}-r\right)}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \quad \text{------ (A)}$$

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Put $r = 1$ in eq (A)

$$\mu_1' = \left(\frac{n_2}{n_1}\right)^1 \frac{\beta\left(\frac{n_1}{2}+1, \frac{n_2}{2}-1\right)}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)}$$

$$= \left(\frac{n_2}{n_1}\right) \frac{\Gamma\left(\frac{n_1}{2}+1\right)\Gamma\left(\frac{n_2}{2}-1\right)}{\Gamma\left(\frac{n_1}{2}+\frac{n_2}{2}\right)} \cdot \frac{\Gamma\left(\frac{n_1}{2}+\frac{n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}$$

$$= \left(\frac{n_2}{n_1}\right) \frac{\frac{n_1}{2}\,\Gamma\left(n_1/2\right)\,\Gamma\left(\frac{n_2}{2}-1\right)}{\Gamma\left(\frac{n_1}{2}\right)\left(\frac{n_2}{2}-1\right)\Gamma\left(\frac{n_2}{2}-1\right)}$$

$$= \frac{n_2}{2\left(\frac{n_2-2}{2}\right)}$$

$$\Rightarrow \mu_1' = \frac{n_2}{n_2-2}$$

Put $r=2$ in eq (A)

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$$\mu_2' = \left(\frac{n_2}{n_1}\right)^2 \cdot \frac{\beta\left(\frac{n_1}{2}+2, \frac{n_2}{2}-2\right)}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)}$$

$$= \left(\frac{n_2}{n_1}\right)^2 \frac{\Gamma\left(\frac{n_1}{2}+2\right)\Gamma\left(\frac{n_2}{2}-2\right)}{\Gamma\left(\frac{n_1+n_2}{2}\right)} \cdot \frac{\Gamma\left(\frac{n_1+n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}$$

$$= \left(\frac{n_2}{n_1}\right)^2 \frac{\left(\frac{n_1}{2}+1\right)\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}-2\right)}{\Gamma\left(\frac{n_1}{2}\right)\left(\frac{n_2}{2}-1\right)\left(\frac{n_2}{2}-2\right)\Gamma\left(\frac{n_2}{2}-2\right)}$$

$$= \frac{n_2^2\, n_1 (n_1+2)}{n_1^2 (n_2-2)(n_2-4)}$$

$$\Rightarrow \quad \mu_2' = \frac{n_2^2 (n_1+2)}{n_1 (n_2-2)(n_2-4)}$$

$$Var(F) = \mu_2' - (\mu_1')^2$$

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$$Var(F) = \frac{n_2^2(n_1+2)}{n_1(n_2-2)(n_2-4)} - \left(\frac{n_2}{n_2-2}\right)^2$$

$$= \frac{n_2^2}{(n_2-2)}\left[\frac{n_1+2}{n_1(n_2-4)} - \frac{1}{n_2-2}\right]$$

$$= \frac{n_2^2}{(n_2-2)}\left[\frac{(n_1+2)(n_2-2) - n_1(n_2-4)}{n_1(n_2-2)(n_2-4)}\right]$$

$$= \frac{n_2^2}{(n_2-2)}\left[\frac{n_1n_2 - 2n_1 + 2n_2 - 4 - n_1n_2 + 4n_1}{n_1(n_2-2)(n_2-4)}\right]$$

$$= \frac{(n_2)^2(2n_1 + 2n_2 - 4)}{n_1(n_2-2)^2(n_2-4)}$$

$\Rightarrow$

$$Var(F) = \frac{2n_2^2(n_1+n_2-2)}{n_1(n_2-2)^2(n_2-4)}$$

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Problem 4) 93, 94, 96, 99, 2000, 97

Relation B/W F & t distribution

Show that the square of a t-variate with n d.f is distributed as F with 1 & n d.f.

Solution:- We have

$$dP(F) = \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} F^{\frac{n_1}{2}-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}} dF \quad 0 \leq F \leq \infty$$

For n_1 & n_2 degree of freedom.

Putting $n_1 = 1$, $n_2 = n$ & $F = t^2 \Rightarrow dF = 2t\,dt$ this yields,

$$dP(t^2) = \frac{\left(\frac{1}{n}\right)^{1/2} (t^2)^{1/2-1}}{\beta\left(\frac{1}{2}, \frac{n}{2}\right)\left(1+\frac{1}{n}t^2\right)^{\frac{1+n}{2}}} 2t\,dt \quad ; 0 \leq t^2 \leq \infty$$

$$dP(t^2) = \frac{1}{\sqrt{n}\,\beta\left(\frac{1}{2}, \frac{n}{2}\right)} t^{-1} \cdot \frac{2t\,dt}{\left(1+\frac{t^2}{n}\right)^{\frac{1+n}{2}}}$$

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$$= \frac{2\,dt}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})} \cdot \frac{1}{(1+\frac{t^2}{n})^{\frac{n+1}{2}}} \quad 0 \le t^2 \le \infty$$

$$dP(t^2) = \frac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})} \left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}} \quad -\infty \le t \le \infty$$

which is t-distribution with n d.f.
Hence t^2-Variate with n d.f is distributed as F with 1 & n d.f.

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1997.

Problem:- Relation b/w χ^2 & F-distribution.

Desire χ^2-Distribution as a limiting form of F-distribution.

OR

Show that as $n_2 \to \infty$, $n_1 F$ tends to be distributed as a χ^2-Variate with n_1 d.f.

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Solution:- The distribution of F with n_1 & n_2 d.f is

$$dP = \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} F^{\frac{n_1}{2}-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}} dF$$

$$dP = \frac{(n_1)^{n_1/2} F^{\frac{n_1}{2}-1}}{(n_2)^{n_1/2}\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}} \frac{\Gamma\frac{n_1+n_2}{2}}{\Gamma\frac{n_1}{2}\Gamma\frac{n_2}{2}} dF$$

$$dP = \frac{(n_1)^{n_1/2} F^{\frac{n_1}{2}-1} dF}{\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}\Gamma\frac{n_1}{2}} \cdot \frac{\Gamma\frac{n_1+n_2}{2}}{\Gamma\frac{n_2}{2}\, n_2^{n_1/2}} \quad (A)$$

Using Stirling's Approximation

$$\Gamma(n+1) = n! = e^{-n} n^{n+\frac{1}{2}} \sqrt{2\pi}$$

$$\frac{\Gamma\frac{n_1+n_2}{2}}{n_2^{\frac{n_1}{2}}\Gamma\frac{n_2}{2}} = \frac{\left(\frac{n_1+n_2}{2}-1\right)!}{n_2^{\frac{n_1}{2}}\left(\frac{n_2}{2}-1\right)!}$$

$$= \frac{e^{-\left(\frac{n_1+n_2}{2}\right)+1}\left[\frac{n_1+n_2}{2}-1\right]^{\frac{n_1+n_2}{2}-1+\frac{1}{2}} \cancel{\sqrt{2\pi}}}{n_2^{\frac{n_1}{2}} e^{-\frac{n_2}{2}+1}\left(\frac{n_2}{2}-1\right)^{\frac{n_2}{2}-1+\frac{1}{2}} \cancel{\sqrt{2\pi}}}$$

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$$= \frac{e^{-\frac{n_1}{2}}\, e^{-\frac{n_2}{2}} \cdot e^{1} \left[\frac{n_1+n_2-2}{2}\right]^{\frac{n_1+n_2}{2}-\frac{1}{2}}}{e^{-\frac{n_2}{2}} \cdot e \cdot n_2^{n_1/2} \left[\frac{n_2-2}{2}\right]^{\frac{n_2}{2}-\frac{1}{2}}}$$

$$= \frac{e^{-\frac{n_1}{2}} \left[\frac{n_2+n_1-2}{2}\right]^{\frac{n_1+n_2}{2}-\frac{1}{2}}}{n_2^{\frac{n_1}{2}} \left[\frac{n_2-2}{2}\right]^{\frac{n_2}{2}-1/2}}$$

$$= \frac{e^{-\frac{n_1}{2}} \left[n_2+n_1-2\right]^{\frac{n_1+n_2}{2}-\frac{1}{2}} \; 2^{\frac{n_2}{2}-1/2}}{n_2^{\frac{n_1}{2}} (n_2-2)^{\frac{n_2}{2}-1/2} \; 2^{\frac{n_1+n_2}{2}-1/2}}$$

$$= \frac{e^{-\frac{n_1}{2}} \left[n_1+n_2-2\right]^{\frac{n_1+n_2}{2}-1/2}}{n_2^{\frac{n_1}{2}} (n_2-2)^{\frac{n_2}{2}-1/2} \; 2^{n_1/2}}$$

$$= \frac{e^{-\frac{n_1}{2}} \cdot n_2^{\frac{n_1+n_2}{2}-1/2} \left[1+\frac{n_1-2}{n_2}\right]^{\frac{n_1+n_2}{2}-1/2}}{n_2^{\frac{n_1}{2}} \; n_2^{\frac{n_2}{2}-\frac{1}{2}} \left(1-\frac{2}{n_2}\right)^{\frac{n_2}{2}-1/2} \; 2^{n_1/2}}$$

$$= \frac{e^{-\frac{n_1}{2}} \left[1+\frac{n_1-2}{n_2}\right]^{\frac{n_1-1}{2}} \left[1+\frac{n_1-2}{n_2}\right]^{n_2/2}}{2^{\frac{n_1}{2}} \left[1-\frac{2}{n_2}\right]^{\frac{n_2}{2}} \cdot \left[1-\frac{2}{n_2}\right]^{-1/2}}$$

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$$= \frac{e^{-\frac{n_1}{2}} \left[1+ \frac{n_1-2}{n_2}\right]^{\frac{n_1-1}{2}} \left[\left(1+\frac{n-2}{n_2}\right)^{n_2}\right]^{1/2}}{2^{\frac{n_1}{2}} \left[\left(1-\frac{2}{n_2}\right)^{n_2}\right]^{1/2} \left[1-\frac{2}{n_2}\right]^{-1/2}}$$

when $n_2 \longrightarrow \infty$

$$\frac{\Gamma\frac{n_1+n_2}{2}}{n_2^{\frac{n}{2}} \Gamma\frac{n_2}{2}} = \frac{e^{-\frac{n_1}{2}} (1) \cdot (e^{n_1-2})^{1/2}}{2^{\frac{n}{2}} (e^{-2})^{1/2} (1)}$$

$$= \frac{1}{2^{n/2}}$$

So A $\Rightarrow$

$$\lim_{n_2\to\infty} dP = \lim_{n_2\to\infty} \frac{n_1^{\frac{n_1}{2}} F^{\frac{n_1}{2}-1}}{\Gamma\frac{n_1}{2} \left[\left(1+\frac{n_1}{n_2}F\right)^{n_2}\right]^{1/2} \left[1+\frac{n_1}{n_2}F\right]^{n_1/2}} dF \cdot \frac{1}{2^{\frac{n_1}{2}}}$$

$$= \frac{n_1^{n_1/2} F^{\frac{n_1}{2}-1}}{\Gamma\frac{n_1}{2} (e^{n_1F})^{1/2} (1)} \frac{1}{2^{n_1/2}} dF$$

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$$\lim_{n \to \infty} dP = \frac{n_1^{n_1/2} F^{\frac{n_1}{2}-1}}{\left|\frac{n_1}{2}\right. e^{\frac{n_1 F}{2}} 2^{\frac{n_1}{2}}} dF$$

Put $F = \frac{\chi^2}{n_1} \Rightarrow n_1 = \frac{\chi^2}{F}$

$$\lim_{n \to \infty} dP = \frac{n_1^{n_1/2} \left(\frac{\chi^2}{n_1}\right)^{\frac{n_1}{2}-1}}{\left|\frac{n_1}{2}\right. e^{\frac{\chi^2}{2}} \cdot 2^{\frac{n_1}{2}}} d\left(\frac{\chi^2}{n_1}\right)$$

$$= \frac{n_1^{\frac{n_1}{2}} (\chi^2)^{\frac{n_1}{2}-1}}{\left|\frac{n_1}{2}\right. e^{\frac{\chi^2}{2}} 2^{\frac{n_1}{2}} n_1^{\frac{n_1}{2}-1}} \cdot \frac{1}{n_1} d(\chi^2)$$

$$= \frac{e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{n_1}{2}-1}}{2^{\frac{n_1}{2}} \left|\frac{n_1}{2}\right.} d\chi^2$$

which is χ^2 - distribution.

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(5 Years Problems)

Problem:- 97, 99 show that the Beta-distribution

$$f(x) = \frac{1}{\beta(\alpha+1, \alpha+1)} x^{\alpha} (1-x)^{\alpha} \quad 0 \le x \le 1$$

becomes the Gamma distribution as the parameter $\alpha \longrightarrow \infty$.

Solution:- $$f(x) = \frac{1}{\beta(\alpha+1, \alpha+1)} x^{\alpha} (1-x)^{\alpha} \quad 0 \le x \le 1$$

$$d f(x) = \frac{1}{\beta(\alpha+1, \alpha+1)} x^{\alpha} (1-x)^{\alpha} \, dx$$

put $y = \alpha x$

$$dx = \frac{dy}{\alpha}$$

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$$d f(y) = \frac{1}{\beta(\alpha+1, \alpha+1)} \left(\frac{y}{\alpha}\right)^{\alpha} \left(1 - \frac{y}{\alpha}\right)^{\alpha} \frac{dy}{\alpha}$$

$$= \frac{\Gamma(\alpha+\alpha+2)}{\Gamma(\alpha+1)\,\Gamma(\alpha+1)} \frac{y^{\alpha}}{\alpha^{\alpha}} \left(1 - \frac{y}{\alpha}\right)^{\alpha} \frac{dy}{\alpha}$$

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$$df = \frac{\Gamma(d+1) + (\alpha+1)}{\Gamma(d+1)} \cdot \frac{1}{\Gamma(\alpha+1)} \frac{y^\alpha}{d^{\alpha+1}} \left(1 - \frac{y}{d}\right)^d dy$$

$$df = (d+1)^{\alpha+1} \cdot \frac{1}{\Gamma(\alpha+1)} \frac{y^\alpha}{d^{\alpha+1}} \left(1 - \frac{y}{d}\right)^d dy$$

$$\because \quad \frac{\Gamma(n+k)}{\Gamma(n)} = n^k$$

Now,

$$\lim_{d \to \infty} df = \lim_{d \to \infty} \frac{d^{\alpha+1}\left(1 + \frac{1}{d}\right)^{\alpha+1}}{\Gamma(\alpha+1)} \cdot \frac{y^\alpha}{d^{\alpha+1}} \left(1 - \frac{y}{d}\right)^d dy$$

$$= \frac{y^\alpha}{\Gamma(\alpha+1)} e^{-y} dy$$

$$\Rightarrow df = \frac{1}{\Gamma(\alpha+1)} e^{-y} y^\alpha dy$$

which is Gamma Distribution.

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96

Problem:- If x is binomally distributed with Parameters n & p what is distribution of $y = n - x$

Solution:-

$$M_y(t) = E(e^{ty})$$

$$= E(e^{t(n-x)})$$

$$= E(e^{tn} \cdot e^{-xt})$$

$$= e^{nt} E(e^{-tx})$$

$$= e^{nt} (q + pe^{-t})^n$$

$$= (e^t q + p)^n$$

$$= (p + qe^t)^n$$

⇒ which is m.g.f of Binomial Distribution.

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Problem:- Find out the Probability that a person tossing 3 coins will get either all heads or all tails for the 2nd time on the 5th toss.

Solution:-

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

let A denotes the Event that all heads occure

$A = \{HHH\}$

$n(A) = 1, \quad n(S) = 8$

$P(A) = \frac{n(A)}{n(S)} = 1/8$

let B denotes the Event that all tails occure

$B = \{TTT\}$

$n(B) = 1, \quad n(S) = 8$

$P(B) = \frac{n(B)}{n(S)} = 1/8$

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$$p = \frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \frac{1}{4}$$

$$q = 1 - p$$

$$q = 1 - \frac{1}{4} = \frac{3}{4}$$

P [3heads OR 3 tails] = $2/8$

$$k = 2, \ x = 5$$

$$P[X = x] = \binom{x-1}{k-1} p^k q^{x-k}$$

$$P[X = 5] = \binom{5-1}{2-1} \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^{5-2}$$

$$= \binom{4}{1} \left(\frac{1}{16}\right) \left(\frac{27}{64}\right)$$

$$= \frac{27}{4 \times 64}$$

$$= 0.1055$$

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Method of Least Square

92, 94, 96, 99

Problem:- Applying Method of least square to solve the system.

$2x + y = 0$

$3x - 2y = 0$

$-x + y = -2$

Solution:- Normal Eq for x

$4x + 2y = 0$

$9x - 6y = 0$

$x - y = 2$

$14x - 5y = 2 \longrightarrow (A)$

Normal Eq for y

$2x + y = 0$

$-6x + 4y = 0$

$-x + y = -2$

$-5x + 6y = -2 \longrightarrow (B)$

$\Rightarrow \quad 5x - 6y = 2 \longrightarrow (B)$

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Solving eq (A) & (B)

$$70x - 25y = 10$$

$$\underset{+}{-}70x \underset{+}{-} 84y = -28$$

$$59y = -18$$

$$y = -18/59$$

$$y = -0.305$$

put in (A)

$$14x - 5(-0.305) = 2$$

$$14x + 1.525 = 2$$

$$x = \frac{2 - 1.525}{14}$$

$$x = 0.034$$

Hence, Required Values are,

$$x = 0.034$$

$$y = -0.305$$

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Problem:- Explain the method of least square, find the most plausible values from the following system.

$$2x + y = 4.8$$

$$x + 3y = 6.3$$

$$3x - 2y = -2.1$$

$$3x + 2y = 8$$

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Solution:- Normal Eq for x

$$4x + 2y = 9.6$$
$$x + 3y = 6.3$$
$$9x - 6y = -6.3$$
$$9x + 6y = 24$$

$$23x + 5y = 33.6 \longrightarrow (A)$$

Normal Eq for y

$$2x + y = 4.8$$
$$3x + 9y = 18.9$$
$$-54x + 36y = 37.8 \Rightarrow$$
$$-6x + 4y = 4.2$$
$$6x + 4y = 16$$

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Adding above Eq we get

$$5x + 18y = 43.9 \longrightarrow (B)$$

Solving Eq (A) & (B).

5×A & 23×B then subtracting we get

$$115x + 25y = 168$$

$$115x + 414y = 1009.7$$

$$-389y = -841.7$$

$$y = 2.164$$

Put in (A) we get

$$x = 0.99$$

Hence,

Required values are,

$$x = 0.99$$

$$y = 2.164$$

As Required.

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Problem:- Explain the method of Least square. How we find the plausible (satisfactory) Values by least square Method.

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OR

Derive Normal Eq By least Square Method.

Solution:- The method of LS, by which curves and Equation are fitted to data, is a method that Consists of minimizing the sum of squares of the differences b/w observed values & Corresponding values obtained from the curve or Equation describing the data.

Finding Plausible values of L.S Method

By the use of LS Method the most satisfactory values of the unknown quantities may be obtained from a number of independent linear Equation in the unknowns when number of Equations are greater than the number of unknowns.

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Suppose that there are "K" Unknowns $x_1, x_2, \ldots, x_K$ & let there are n Eq where $n > K$

i.e,

$$a_1x_1 + b_1x_2 + \ldots + f_1x_K = l_1$$

$$a_2x_1 + b_2x_2 + \ldots + f_2x_K = l_2$$

$$\ldots$$

$$a_nx_1 + b_nx_2 + \ldots + f_nx_K = l_n$$

where a's, b's, ..., l's are constants.

The least square method calls for the selection of these values of $x_1, x_2, \ldots, x_K$ which make the sum of squares of differences D_i's also called Errors or residuals a minimum, where,

$$D_i = a_ix_1 + b_ix_2 + \ldots + f_ix_K - l_i$$

$$i = 1, 2, \ldots, n$$

$$S = \sum_{i=1}^{n} D_i^2$$

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$$S = \sum_{i=1}^{n} \left[a_i x_1 + b_i x_2 + \cdots + f_i x_k - l_i\right]^2$$

It is obviously that $S = f(x_1, x_2, \ldots, x_k)$

For min value of x

$$\frac{\partial S}{\partial x_1} = \frac{\partial S}{\partial x_2} = \frac{\partial S}{\partial x_3} = \cdots = \frac{\partial S}{\partial x_k} = 0$$

$$\frac{\partial S}{\partial x_1} = 2 \sum_{i=1}^{n} (a_i x_1 + b_i x_2 + \cdots + f_i x_k - l_i)\, a_i = 0$$

$$\frac{\partial S}{\partial x_2} = 2 \sum_{i=1}^{n} (a_i x_1 + b_i x_2 + \cdots + f_i x_k - l_i)\, b_i = 0$$

— — — — — — — — — — — —

— — — — — — — — — — — —

$$\frac{\partial S}{\partial x_k} = 2 \sum_{i=1}^{n} (a_i x_1 + b_i x_2 + \cdots + f_i x_k - l_i)\, f_i = 0$$

The above Eq/s may be written as,

$$x_1 \Sigma a_i^2 + x_2 \Sigma a_i b_i + \cdots + x_k \Sigma a_i f_i = \Sigma a_i l_i$$

$$x_1 \Sigma a_i b_i + x_2 \Sigma b_i^2 + \cdots + x_k \Sigma b_i f_i = \Sigma b_i l_i$$

— — — — — — — — — — — —

— — — — — — — — — — — —

$$x_1 \Sigma a_i f_i + x_2 \Sigma b_i f_i + \cdots + x_k \Sigma f_i^2 = \Sigma f_i l_i$$

The above Eqs are called Normal Eq for $x_1, x_2, \ldots, x_k$ respectively.

Solving these Normal Eqs we get the most plausible values of the system.

Remark*:- It should be noted that the Normal Eq for a set of variables are obtained by multiplying co-efficient of respective variable in the Eq & then adding them togeather.

Problem:- Applying L.S Method (By def) to solve the system,

$$2x+y=0$$

$$3x-2y=0$$

$$-x+y=-2$$

Solution:- Let S be sum of squares of residues (Error)

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Problem :- Applying L.S. Method (By def) to solve the system,

$$x - y - 3 = 0$$

$$3x + 2y - 4 = 0$$

$$2x - 3y + 1 = 0$$

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Solution :-

$$S = (x-y-3)^2 + (3x+2y-4)^2 + (2x-3y+1)^2$$

$$\frac{\partial S}{\partial x} = 2(x-y-3) + 2(3x+2y-4)3 + 2(2x-3y+1)(2)$$

Put $\frac{\partial S}{\partial x} = 0$

$$2[x - y - 3 + 9x + 6y - 12 + 2 + 4x - 6y] = 0$$

$$14x - y - 13 = 0$$

$$14x - y = 13 \longrightarrow (A)$$

$$\frac{\partial S}{\partial y} = 2(x-y-3)(-1) + 2(3x+2y-4)(2) + 2(2x-3y+1)(-3)$$

Put $\frac{\partial S}{\partial y} = 0$

$$2[-x + y + 3 + 6x + 4y - 8 - 6x + 9y - 3] = 0$$

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$S = (2x+y-0)^2 + (3x-2y-0)^2 + (-x+y+2)^2$

$\frac{\partial S}{\partial x} = 2(2x+y)2 + 2(3x-2y)3 + 2(-x+y+2)(-1)$

put $\frac{\partial S}{\partial x} = 0$

$2\left[4x+2y+9x-6y+x-y-2\right] = 0$

$14x - 5y = 2 \longrightarrow (A)$

Now,

$\frac{\partial S}{\partial y} = 2(2x+y) + 2(3x-2y)(-2) + 2(-x+y+2)$

put $\frac{\partial S}{\partial y} = 0$

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$\Rightarrow 2\left[2x+y+4y-6x-x+y+2\right] = 0$

$-5x+6y = -2$

$5x-6y = 2 \longrightarrow (B)$

solving Eq (A) & (B) we get

$x = 0.034$

$y = -0.305$

As Required.

$-x + 14y = 8$

$x - 14y = -8 \longrightarrow$ (B)

Multiplying Eq (A) by 14 & then subtracting from (B)

$x - 14y = -8$

$\underline{\oplus}(14)^2 x \ominus 14y = \underline{\oplus}(13)(14)$

$-195x = -190$

$x = \frac{190}{195}$

$\boxed{x = 0.974}$

Put in (B)

$0.974 - 14y = -8$

$14y = 0.974 + 8$

$y = \frac{8.0974}{14}$

$\boxed{y = 0.641}$

As Required.

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"Curve Fitting"

Some times the relationship b/w two or more variables exists for example the height of the children depends on their age, we can express such relationship in the form of an Eq. If x & y are two variables we can express their relation in different form,

$$y = a + bx$$

$$y = a + bx + cx^2$$

$$y = a e^{bx}$$

$$y = a b^x$$

where a, b & c are constants.

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Problem:- Fit a st line by least square method in the following data

x :	1	2	3	4	5
y :	3	4	6	9	10

Also find square of deviation.

Solution:- let $y = a + bx$

be the Eq of straight line, For Normal Equations,

$$\Sigma y = \Sigma a + b\Sigma x$$

$$\Sigma y = na + b\Sigma x \rightarrow (1)$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 \rightarrow (2)$$

x	y	xy	x^2	$\hat{y}=a+bx$	$y-\hat{y}$	$(y-\hat{y})^2$
1	3	3	1	2.6	0.4	0.16
2	4	8	4	4.5	-0.5	0.25
3	6	18	9	6.4	-0.4	0.16
4	9	36	16	8.3	0.7	0.49
5	10	50	25	10.3	-0.3	0.9
15	32	115	55			

So Eq (1) & (2) becomes,

$$32 = 5a + 15b \longrightarrow (3)$$

$$115 = 15a + 55b \longrightarrow (4)$$

solving Eq (3) & (4)

X-ing Eq (3) by 5 & then subtracting from (4)

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$$15a + 55b = 115$$
$$\textcircled{-} 15a \textcircled{-} 45b = \textcircled{-} 96$$

$$10b = 19$$

$$\boxed{b = 1.9}$$

put in ③

$$5a + 15(1.9) = 32$$

$$a = \frac{3.5}{5}$$

$$\boxed{a = 0.7}$$

So, Required straight line is,

$$y = 0.7 + 1.9x$$

$$y = \frac{0.7}{10} + \frac{1.9}{10}x$$

$$10y = 7 + 19x$$

Also,

$$\sum (y - \hat{y})^2 = 1.96$$

As Required.

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Problem:- Find 2nd degree Parabola to the following data.

x: 0 1 2 3 4

y: 1 1.8 1.3 2.5 6.3.

Solution:- let

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$$y = a + bx + cx^2$$

be the required 2nd degree Parabola,

$$\Sigma y = na + b\Sigma x + c\Sigma x^2 \rightarrow (1)$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3 \rightarrow (2)$$

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 \rightarrow (3)$$

x	y	x^2	x^3	x^4	xy	x^2y
0	1	0	0	0	0	0
1	1.8	1	1	1	1.8	1.8
2	1.3	4	8	16	2.6	5.2
3	2.5	9	27	81	7.5	22.5
4	6.3	16	64	256	25.2	100.8
10	12.9	30	100	354	37.1	130.3

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Eq ①, ② & ③ becomes,

$5a + 10b + 30c = 12.9 \rightarrow ③$

$10a + 30b + 100c = 37.1 \rightarrow ④$

$30a + 100b + 354c = 130.3 \rightarrow ⑤$

X—ing Eq ③ by 2 & then subtracting from eq ④

$10a + 30b + 100c = 37.1$

$\pm 10a \pm 20b \pm 60c = \pm 25.8$

$10b + 40c = 11.3$

$b + 4c = 1.13 \rightarrow ⑥$

X—ing Eq ③ by 6 then subtracting from Eq ⑤

$30a + 100b + 354c = 130.3$

$\pm 30a \pm 60b \pm 180c = \pm 77.4$

$40b + 174c = 52.9$

$b + 4.35c = 1.3225 \rightarrow ⑦$

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Subtracting Eq ⑥ & ⑦ we get

$$b + 4.35c = 1.3225$$
$$\underline{-b - 4c = -1.13}$$

$$0.35c = 0.1925$$

$$c = \frac{0.1925}{0.35}$$

$$\boxed{c = 0.55}$$

Put in ⑥

$$b + 4(0.55) = 1.13$$

$$b = 1.13 - 2.2$$

$$\boxed{b = -1.07}$$

Put the values of b & c in eq ③ we get

$$5a + 10(-1.07) + 30(0.55) = 12.9$$

$$5a - 10.7 + 16.5 = 12.9$$

$$5a = 12.9 + 10.7 - 16.5$$

$$a = \frac{7.1}{5} = 1.42$$

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So Required Parabola is,

$y = 1.42 - 1.07x + 0.55x^2$

As Required.

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Problem:- Fit a 2nd degree Parabola for the following data.

X :	0	2	4	6	8	10	12
Y :	1.0	5.0	10.2	22.6	38.5	42.6	47.4

Solution:- Let $y = a + bx + cx^2$ be the required 2nd degree Parabola,

$$\Sigma y = na + b\Sigma x + c\Sigma x^2 \longrightarrow (1)$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3 \longrightarrow (2)$$

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 \longrightarrow (3)$$

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X	Y	X2	X3	X4	XY	X²Y
0	1	0	0	0	0	0
2	5	4	8	16	10	20
4	10.2	16	64	256	40.8	163.2
6	22.6	36	216	1296	135.6	813.6
8	38.5	64	512	4096	308	2464
10	42.6	100	1000	10000	426	4260
12	47.4	144	1728	20,736	568.8	6825.6
42	167.3	364	3528	36400	1489.2	145464

So Eq ①, ② & ③ Takes form,

$167.3 = 7a + 42b + 364c \longrightarrow ④$

$1489.2 = 42a + b(364) + c(3528) \longrightarrow ⑤$

$14546.4 = 364a + 3528b + 36400c \longrightarrow ⑥$

$\Rightarrow \quad a + 6b + 52c = 23.9 \longrightarrow ⑦$

$a + 8.67b + 84c = 35.45 \longrightarrow ⑧$

$a + 9.69b + 100c = 39.97 \longrightarrow ⑨$

subtracting ⑦ from ⑧ &

then subtracting from ⑨

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$$a + 8.67b + 84c = 35.45$$
$$\ominus a \ominus 6b \ominus 52c = \ominus 23.9$$

$$2.67b + 32c = 10.55$$

$$b + 11.99c = 3.95 \rightarrow (A)$$

$$a + 9.69b + 100c = 39.97$$
$$\ominus a \ominus 6b \ominus 52c = \ominus 23.9$$

$$3.69b + 48c = 16.07$$

$$b + 13c = 4.35 \rightarrow (B)$$

Subtracting eq (A) from (B)

$$b + 13c = 4.35$$
$$\ominus b \ominus 11.99c = \ominus 3.95$$

$$1.01c = 0.4$$

$$\boxed{c = 0.40}$$

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put in. (A)

$b + 11.99(0.40) = 3.95$

$b = 3.95 - 4.796$

$\boxed{b = 0.84}$

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& put the values of a, b & c in eq (P) we get

$a + 6(0.84) + 52(0.40) = 23.9$

$a + 5.07 + 20.8 = 23.9$

$a = 23.9 - 25.87$

$\boxed{a = -1.97}$

So, Required 2nd degree parabola is,

$y = -1.97 + 0.84x + 0.40x^2$

As Required.

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Problem:- Fit a 2nd degree curve to the following data.

X :	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Y :	1.1	1.8	1.6	2.0	3.7	1.4	3.5

Solution:- let $y = a + bx + cx^2$ be the required 2nd degree curve &

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

X	Y	X^2	X^3	X^4	XY	X^2Y
1.0	1.1	1	1	1	1.1	1.1
1.5	1.8	2.25	3.375	5.0625	2.7	4.05
2.0	1.6	4	8	16	3.2	6.4
2.5	2.0	6.25	15.625	39.0625	5	12.5
3.0	3.7	9	27	81	11.1	33.3
3.5	1.4	12.25	42.875	150.0625	4.9	17.15
4.0	3.5	16	64	256	14	56
17.5	15.1	50.75	161.875	548.1875	42	130.5

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So above Eq Takes the form,

$$15.1 = 7a + 17.5b + 50.75c$$

$$42 = 17.5a + 50.75b + 161.875c$$

$$130.5 = 50.75a + 161.875b + 548.1875c$$

$\Rightarrow$

$$a + 2.5b + 7.25c = 2.15 \rightarrow (1)$$

$$a + 2.9b + 9.25c = 2.4 \rightarrow (2)$$

$$a + 3.18b + 10.80c = 2.57 \rightarrow (3)$$

Subtracting Eq (1) from Eq (2) & Eq (3) respectively we get,

$$a + 2.9b + 9.25c = 2.4$$

$$\underline{\pm a \pm 2.5b \pm 7.25c = \pm 2.15}$$

$$0.4b + 2c = 0.25$$

$$\Rightarrow \quad b + 5c = 0.625 \rightarrow (4)$$

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$$a + 3.18b + 10.80c = 2.57$$

$$\overset{-}{\oplus} a \overset{-}{\oplus} 2.5b \overset{-}{\oplus} 7.25c = \overset{-}{\oplus} 2.15$$

$$0.68b + 3.55c = 0.42$$

$$b + 5.22c = 0.61 \rightarrow (5)$$

Subtracting eq. (4) from (5) we get

$$b + 5.22c = 0.61$$

$$\overset{-}{\oplus} b \overset{-}{\oplus} 5c = \overset{-}{\oplus} 0.625$$

$$0.22c = -0.015$$

$$\boxed{c = -0.06}$$

Put in (4)

$$b + 5(-0.06) = 0.625$$

$$b = 0.625 + 0.3$$

$$\boxed{b = 0.925}$$

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put in (1) we get

$a + 2.5(0.925) + 7.25(-0.06) = 2.15$

$a + 2.3125 - 0.435 = 2.15$

$a = 0.2725$

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So Required curve is,

$y = 0.2725 + 0.925x - 0.06x^2$

As Required.

Problem:- Fit an Exponential curve $y = ae^{bx}$ to the following data.

x:	1	2	3	4	5	6
y:	1.6	4.5	13.8	40.2	125.0	363

Solution:- Given

$y = ae^{bx}$

$\log y = \log a + bx \log e$

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$Y = A + Bx$

where

$Y = \log y$

$A = \log a$

$B = b \log e$

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Normal Eqs are,

$\sum Y = nA + B\sum x$

$\sum xY = A\sum x + B\sum x^2$

x	y	$Y = \log y$	x^2	xY
1	1.6	0.2041	1	0.2041
2	4.5	0.6532	4	1.3064
3	13.8	1.1399	9	3.4197
4	40.2	1.6042	16	6.4168
5	125.0	2.0969	25	10.4845
6	363	2.5599	36	15.3594
21		8.2582	91	37.2449

So above Eq becomes,

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$6A + 21B = 8.2582 \longrightarrow (1)$

$21A + 91B = 37.2449 \longrightarrow (2)$

×-ing eq (1) by (7) & eq (2) by (2)

& then subtracting we get

$42A + 182B = 74.4998$

$\ominus 42A \oplus 147B = \ominus 57.8074$

$35B = 16.6924$

$B = 0.4769$

put in (1)

$6A + 21(0.4769) = 8.2582$

$6A = 8.2582 - 10.0149$

$A = \frac{-1.7567}{6}$

$A = -0.2927$

$\Rightarrow \quad a = \text{Antilog } A = \text{Antilog}(-0.2927)$

$\boxed{a = 0.50}$

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$0.4769 = b.(0.4343)$

$$b = \frac{0.4769}{0.4343}$$

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$$\boxed{b = 1.09}$$

So,

$$y = 0.50\, e^{1.09x}$$

Required line.

Problem:- Fit an Eq of the form $y = ax^b$ to the following

x:	1	2	3	4	5	6
y:	2.98	4.26	5.21	6.10	6.80	7.50

Solution:- Given

$$y = ax^b$$

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$\log y = \log a + b \log x$

$Y = A + bX$

$Y = \log y$

$A = \log a$

$X = \log x$

Normal eq's are

$\sum Y = nA + b\sum X$

$\sum XY = A\sum X + b\sum X^2$

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x	y	X = log x	Y = log y	X^2	XY
1	2.98	0	0.4742	0	0
2	4.26	0.3010	0.6294	0.0906	0.1894
3	5.21	0.4771	0.7168	0.2276	0.3420
4	6.10	0.6021	0.7853	0.3625	0.4728
5	6.80	0.6990	0.8325	0.4886	0.5819
6	7.50	0.7782	0.875	0.6056	0.6809
		2.8574	4.3133	1.7749	2.2670

So, above Eq becomes,

$$6A + 2.8574b = 4.3133$$

$$2.8574A + 1.7749b = 2.2670$$

$$\Rightarrow \quad A + 0.4762b = 0.7189 \longrightarrow (1)$$

$$\ominus A \ominus 0.6212b = \ominus 0.7934 \longrightarrow (2)$$

$$-0.1450b = -0.0745$$

$$\Rightarrow b = 0.5138$$

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Put in (1)

$$A + 0.4762(0.5138) = 0.7189$$

$$A = 0.4742$$

$$a = \text{Antilog} A$$

$$a = 2.9801$$

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So, Required curve is,

$$y = 2.9801(x)^{0.5138}$$

As Required.

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Regression

The relation between the Expected value of the dependent variable and the independent variable is called a Regression Relation. When we study the dependence of a variable on a single independent variable, it is called a simple or two variable regression. When dependence of a variable on two or more than two independent variables is studied, it is called multiple regression. Furthermore when the dependence is represented by a straight line Eq it is said to linear, otherwise it is said to be curvilinear.

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Regression Equation Y on X

The Regression Eq taking x as independent Variable and y as dependent Variable is,

$$y = a + bx$$

where a & b are unknown constants to be determined by using Principle of least Squares

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The Normal Eqs are

$$\Sigma y = na + b\Sigma x \rightarrow (1)$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 \rightarrow (2)$$

X-ing Eq (1) by Σx & (2) by n we get

$$\Sigma x \Sigma y = na\Sigma x + b(\Sigma x)^2$$

$$n\Sigma xy = na\Sigma x + nb\Sigma x^2$$

Subtracting we get

$$n\Sigma xy - \Sigma x\Sigma y = b\left[n\Sigma x^2 - (\Sigma x)^2\right]$$

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}$$

OR

$$b = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sum (x-\bar{x})^2}$$

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Similarly we get

$$a = \frac{\sum x^2 \sum y - \sum x \sum xy}{n\sum x^2 - (\sum x)^2}$$

As Required.

Correlation:-

Correlation closely related with regression is conceptually quite different from it. Regression is a relation of dependence but Correlation is a measure of the strength

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or degree of relationship or the interdependence between two variables.

Two variables are said to be correlated if they tend to simultaniously vary in some direction. If both the variables tend to increase (or decrease), the correlation is said to be direct or positive. If the variables tend to increase (or decrease) in opposite direction, then correlation is said to be negative.

Correlation Co-efficients

The correlation co-efficients is denoted by "r" & is calculated as,

$$r_{xy} = \frac{\Sigma(x-\bar{x})\,\Sigma(y-\bar{y})}{\sqrt{\Sigma(x-\bar{x})^2\,\Sigma(y-\bar{y})^2}}$$

OR

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$$r_{xy} = \frac{\sum xy - \frac{(\sum x)(\sum y)}{n}}{\sqrt{\left(\sum x^2 - \frac{(\sum x)^2}{n}\right)\left(\sum y^2 - \frac{(\sum y)^2}{n}\right)}}$$

OR

$$r_{xy} = \frac{Cov(x, y)}{\sqrt{Var\,x \cdot Var\,y}}$$

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Properties of "r":

The Correlation Co-efficient r has the following properties,

i) The Correlation Co-efficient "r" is symmeterical w.r.t variables x & y i.e.,

$$r_{xy} = r_{yx}$$

ii) The Correlation Co-efficient lies b/w -1 & +1 i.e.,

$$-1 \leq r \leq +1$$

iii) The Correlation Co-efficient is independent of the origin & scale,

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Problem:- Calculate Correlation Co-efficient b/w X & Y from the following data,

X :	1	2	3	4	5
Y :	2	5	3	8	7

Solution:- we use the relation

$$\gamma_{xy} = \frac{\Sigma (x-\bar{x}) \cdot (y-\bar{y})}{\sqrt{\Sigma (x-\bar{x})^2 \ \Sigma (y-\bar{y})^2}}$$

X	Y	$x-\bar{x}$	$(x-\bar{x})^2$	$y-\bar{y}$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
1	2	-2	4	-3	9	6
2	5	-1	1	0	0	0
3	3	0	0	-2	4	0
4	8	1	1	3	9	3
5	7	2	4	2	4	4
Σ=15	25	0	10	0	26	13

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$\bar{X} = \frac{\Sigma X}{n} = \frac{15}{5} = 3$

$\bar{Y} = \frac{\Sigma Y}{n} = \frac{25}{5} = 5$

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So,

$$r_{xy} = \frac{13}{\sqrt{10 \times 26}} = \frac{13}{\sqrt{260}}$$

$$\boxed{r_{xy} = 0.8}$$

We can also calculated $r_{x.y}$ using alternative form i.e,

$$r_{xy} = \frac{\Sigma XY - (\Sigma X)(\Sigma Y)/n}{\sqrt{\left[\Sigma X^2 - \frac{(\Sigma X)^2}{n}\right]\left[\Sigma Y^2 - \frac{(\Sigma Y)^2}{n}\right]}}$$

X	Y	X^2	Y^2	XY
1	2	1	4	2
2	5	4	25	10
3.	3	9	9	9
4	8	16	64	32
5	7	25	49	35
15	25	55	151	88

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So,

$$\gamma_{xy} = \frac{88 - (15)(25)/5}{\sqrt{\left[55 - \frac{(15)^2}{5}\right]\left[151 - \frac{(25)^2}{5}\right]}}$$

$$\gamma_{xy} = \frac{88 - 75}{\sqrt{(55-45)\,(151-125)}}$$

$$\gamma_{xy} = \frac{13}{\sqrt{10 \times 26}} = \frac{13}{\sqrt{260}}$$

$$\gamma_{xy} = 0.8$$

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Problem 2001:- The data have been obtained on X = rainfall (inches)
Y = yield of wheat

X :	12.9	7.2	11.3	18.6	8.8	10.3	15.9	13.1	12.5	14.0
Y :	62.5	28.7	52.2	80.6	41.6	44.5	71.3	54.4	53.1	55.3

i) Fit a least square line to predict yield of wheat.

ii) Obtain simple correlation co-efficient & interpret.

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Solution:- i) let

$y = a + bx$ be the least square line,

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Normal Eqs are

$\sum y = na + b\sum x \longrightarrow (1)$

$\sum xy = a\sum x + b\sum x^2 \longrightarrow (2)$

x	y	x^2	y^2	xy
12.9	62.5	166.41	3906.25	806.25
7.2	28.7	51.84	823.69	206.64
11.3	52.2	127.69	2724.84	589.86
18.6	80.6	345.96	6496.36	1499.16
8.8	41.6	77.44	1730.56	366.08
10.3	44.5	106.09	1980.25	458.35
15.9	71.3	252.81	5083.69	1133.67
13.1	54.4	171.61	2959.36	712.64
12.5	53.1	156.25	2819.61	663.75
14.0	55.3	196	3058.09	774.2
124.6	544.2	1652.1	31582.7	7210.6

So Eq (1) & (2) becomes

$10a + 124.6b = 544.2$

$124.6a + 1652.1b = 7210.6$

$\Rightarrow$

$a + 12.46b = 54.42 \rightarrow (3)$

$a + 13.25b = 57.86 \rightarrow (4)$

Subtracting eq (3) from eq (4) we get

$a + 13.25b = 57.86$

$\underline{\pm a \pm 12.46b = \pm 54.42}$

$0.79b = 3.44$

$\boxed{b = 4.35}$

put in (3)

$a + 12.46(4.35) = 54.42$

$a = 54.42 - 54.201$

$\boxed{a = 0.219}$

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So, Required least square line is,

$y = 0.219 + 4.35x$

As Required.

ii) $\gamma_{xy} = \dfrac{\Sigma xy - (\Sigma x)(\Sigma y)/n}{\sqrt{\left[\Sigma x^2 - \frac{(\Sigma x)^2}{n}\right]\left[\Sigma y^2 - \frac{(\Sigma y)^2}{n}\right]}}$

$\gamma_{xy} = \dfrac{7210.6 - (124.6)(544.2)/10}{\sqrt{\left[1652.1 - \frac{(124.6)^2}{10}\right]\left[31582.7 - \frac{(544.2)^2}{10}\right]}}$

$\gamma_{xy} = \dfrac{7210.6 - 6780.732}{\sqrt{(99.584)(1967.336)}}$

$\gamma_{xy} = \dfrac{429.868}{442.623}$

$\gamma_{xy} = 0.97$ As Required.

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Problem:- Distinguish b/w Correlation & Regression & also find the Co-efficient of correlation b/w X & Y.

X : 4 7 8 6 5

Y : 10 12 15 18 13

Solution:-

X	Y	X^2	Y^2	XY
4	10	16	100	40
7	12	49	144	84
8	15	64	225	120
6	18	36	324	108
5	13	25	169	65
30	68	190	962	417

$$r_{xy} = \frac{\Sigma XY - (\Sigma X)(\Sigma Y)/n}{\sqrt{\left(\Sigma x^2 - \frac{(\Sigma x)^2}{n}\right)\left(\Sigma y^2 - \frac{(\Sigma y)^2}{n}\right)}}$$

$$r_{xy} = \frac{417 - (30)(68)/5}{\left[\left(190 - \frac{(30)^2}{5}\right)\left(962 - \frac{(68)^2}{5}\right)\right]^{1/2}}$$

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$$= \frac{417 - 408}{\sqrt{(10)(37.2)}}$$

$$= \frac{9}{\sqrt{372}}$$

$$= \frac{9}{19.28}$$

$= 0.47$ As Required.

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Problem:-
Determine the trend line by least square method.

Year :-	1945	1946	1947	1948	1949	1950	1951	1952	1953
Price :-	3	6	2	10	7	9	14	12	18

Solution:- let the Eq of the linear trend be

$$y = a + bx$$

Normal Eqs are

$$\Sigma y = na + b\Sigma x \rightarrow (1)$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 \rightarrow (2)$$

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Year	x	y	x^2	xy
1945	-4	3	16	-12
1946	-3	6	9	-18
1947	-2	2	4	-4
1948	-1	10	1	-10
1949	0	7	0	0
1950	1	9	1	9
1951	2	14	4	28
1952	3	12	9	36
1953	4	18	16	72
	0	80	60	101

So, Eq ① & ② Takes the Form

$$9a + b(0) = 80$$

$$a(0) + b(60) = 101$$

$\Rightarrow$ $9a = 80 \Rightarrow a = 80/9 = 8.88$

$\Rightarrow$ $b = \frac{101}{60} = 1.68$

So, Required line is,

$$y = 8.88 + 1.68x$$

As Required.

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Problem:- Fit a 2nd degree trend curve (Parabola) to the following data & compute the trend value.

Year :-	1931	1933	1935	1937	1939	1941	1943	1945
Prices:-	96	87	91	102	108	139	307	289

Solution:- let the Eq of 2nd degree trend curve be,

$$y = a + bx + cx^2$$

Normal Eqs are

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

Year	x	y	x^2	x^3	x^4	xy	x^2y
1931	-7	96	49	-343	2401	-672	4704
1933	-5	87	25	-125	625	-435	2175
1935	-3	91	9	-27	81	-273	819
1937	-1	102	1	-1	+1	-102	102
1939	1	108	1	1	1	108	108
1941	3	139	9	27	81	417	1251
1943	5	307	25	125	625	1535	7675
1945	7	289	49	343	2401	2023	14161
	0	1219	168	0	6216	2601	30995

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So above Eq becomes

$8a + 168c = 1219$

$168b = 2601$

$\Rightarrow \boxed{b = 15.48}$

$168a + 6216c = 30995$

$0.02a + c = 4.98 \rightarrow (3)$

$0.04a + c = 7.25 \rightarrow (4)$

Subtracting Eq (3) from eq (4)

$$\begin{aligned} 0.04a + c &= 7.25 \\ -0.02a - c &= -4.98 \\ \hline 0.02a &= 2.27 \end{aligned}$$

$a = \frac{2.27}{0.02}$

$\boxed{a = 113.5}$

Put in (3)

$(113.5)(0.02) + c = 4.98$

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$c = 4.98 - 2.27$

$c = 2.71$

So Required curve is,

$y = 113.5 + 15.48x + 2.71x^2$

As Required.

93, 94, 95, 2000

Problem:- Calculate all the Partial regression & Partial correlation co-efficients.

$\sigma_1 = 2.8, \quad \sigma_2 = 1.2, \quad \sigma_3 = 1.5$

$r_{12} = 0.75, \quad r_{23} = 0.54, \quad r_{31} = 0.43$

Solution:- Partial regression Co-efficents are $b_{12.3}, b_{13.2}, b_{23.1}$

&

Partial Correlation Co-efficients are $r_{12.3}, r_{13.2}, r_{23.1}$

$$b_{12.3} = \frac{r_{12} - r_{13} r_{23}}{1 - r_{23}^2} \times \frac{\sigma_1}{\sigma_2}$$

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$$b_{12\cdot 3} = \frac{0.75 - (0.43)(0.54)}{1-(0.54)^2} \times \frac{2.8}{1.2}$$

$$b_{12\cdot 3} = 1.70553$$

$$b_{13\cdot 2} = \frac{r_{13} - r_{12}\, r_{32}}{1 - r_{32}^2} \times \frac{\sigma_1}{\sigma_3}$$

$$= \frac{0.43 - (0.75)(0.54)}{1-(0.54)^2} \times \frac{2.8}{1.5}$$

$$= 0.065876$$

$$b_{23\cdot 1} = \frac{r_{23} - r_{21}\, r_{31}}{1 - r_{31}^2} \times \frac{\sigma_2}{\sigma_3}$$

$$= \frac{0.54 - (0.75)(0.43)}{1-(0.43)^2} \times \frac{1.2}{1.5}$$

$$b_{23\cdot 1} = 0.213470$$

Now we calculate Partial Correlation co-efficients :-

$$r_{12\cdot 3} = \frac{r_{12} - r_{13}\, r_{23}}{\sqrt{1-r_{13}^2}\,\sqrt{1-r_{23}^2}}$$

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$$r_{12\cdot3} = \frac{0.75 - (0.43)(0.54)}{\sqrt{1-(0.43)^2}\sqrt{1-(0.54)^2}}$$

$$r_{12\cdot3} = 0.46823$$

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$$r_{23\cdot1} = \frac{r_{23} - r_{21}r_{31}}{\sqrt{1-r_{21}^2}\sqrt{1-r_{31}^2}} = \frac{0.54 - (0.75)(0.43)}{\sqrt{1-(0.75)^2}\sqrt{1-(0.43)^2}}$$

$$r_{23\cdot1} = 0.36422$$

$$r_{13\cdot2} = \frac{r_{13} - r_{12}r_{32}}{\sqrt{1-r_{12}^2}\sqrt{1-r_{32}^2}} = \frac{0.43 - (0.75)(0.54)}{\sqrt{1-(0.75)^2}\sqrt{1-(0.54)^2}}$$

$$r_{13\cdot2} = 0.03779$$

As Required.

Wish You Good Luck!

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Problem:- Given

$$r_{12} = 0.492$$

$$r_{13} = 0.927$$

$$r_{23} = 0.758$$

Find all the Partial correlation co-efficients.

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Solution :-

$$r_{12\cdot3} = \frac{r_{12} - r_{13}r_{23}}{\sqrt{1-r_{13}^2}\sqrt{1-r_{23}^2}}$$

$$= \frac{0.492 - (0.927)(0.758)}{\sqrt{1-(0.927)^2}\sqrt{1-(0.758)^2}}$$

$$= \frac{0.492 - 0.702666}{(0.7790)(0.6522)}$$

$$= \frac{-0.210}{0.508} = -0.413$$

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$$r_{23.1} = \frac{r_{23} - r_{21} r_{31}}{\sqrt{1 - r_{21}^2}\sqrt{1 - r_{31}^2}}$$

$$= \frac{0.758 - (0.492)(0.927)}{\sqrt{1-(0.492)^2}\sqrt{1-(0.927)^2}}$$

$$= \frac{0.758 - 0.456}{(0.870)(0.375)}$$

$$= \frac{0.302}{0.326}$$

$$r_{23.1} = 0.926$$

$$r_{31.2} = \frac{r_{31} - r_{32} r_{12}}{\sqrt{1 - r_{32}^2}\sqrt{1 - r_{12}^2}}$$

$$= \frac{0.927 - (0.758)(0.492)}{\sqrt{1-(0.758)^2}\sqrt{1-(0.492)^2}}$$

$$= \frac{0.927 - 0.372}{(0.652)(0.870)}$$

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$\gamma_{3.2} = \frac{0.555}{0.567}$

$\gamma_{3.2} = 0.978$

As Required

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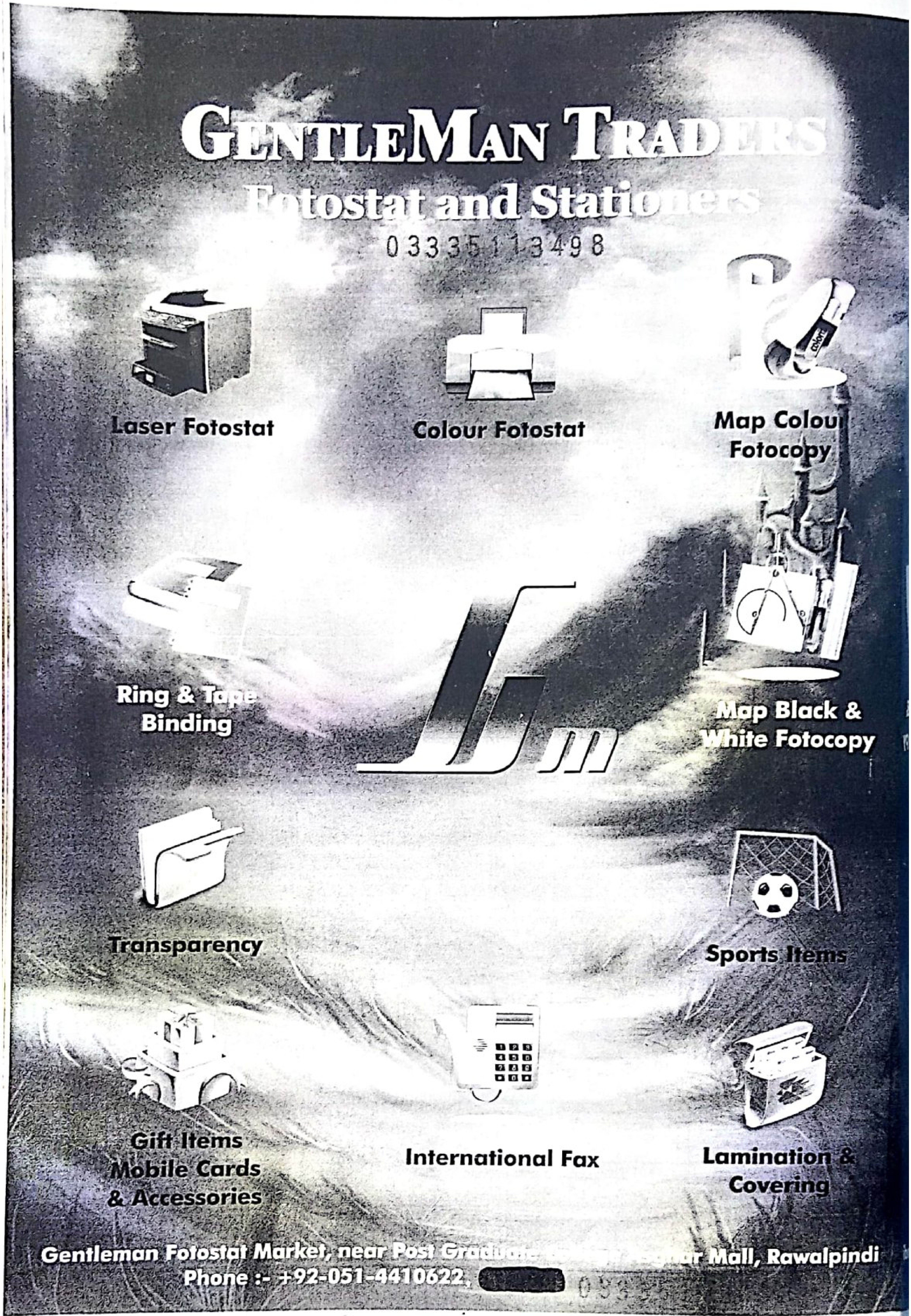

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